18.022 Practice Problems, 11/25/2013

Recitation Instructor: Homer Reid

1.

For each of the following pairings of a map $\mathbf{X} : \mathbb{R}^2 \to \mathbb{R}^3$ and a region $D \in \mathbb{R}^2$, **(a)** describe the surface $\mathbf{X}(D) \cup \mathbb{R}^3$ (that is, the two-dimensional subset of $\mathbb{R}^3$ obtained as the image of D under $\mathbf{X}$), **(b)** compute the standard normal vector $\mathbf{T}(s,t)$ to the surface, **(c)** write down a two-dimensional integral expression for the surface area. (If the surface is infinite, write an expression for the surface area of the portion lying between the planes $z = \pm 10$.) In this problem, a and b are constant parameters and $a > b$.

(a)

$$\mathbf{X}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix} = \begin{pmatrix} a\cos s \\ a\sin s \\ t \end{pmatrix}, \qquad D = \left\{ \begin{array}{c} 0 \le s \le 2\pi \\ -\infty \le t \le \infty. \end{array} \right\}$$

(b)

$$\mathbf{X}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix} = \begin{pmatrix} t\cos s \\ t\sin s \\ t \end{pmatrix}, \qquad D = \left\{ \begin{array}{c} 0 \le s \le 2\pi \\ -\infty \le t \le \infty. \end{array} \right\}$$

(c)

$$\mathbf{X}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix} = \begin{pmatrix} a\cos s\sin t \\ a\sin s\sin t \\ a\cos s \end{pmatrix}, \qquad D = \left\{ \begin{array}{c} 0 \le s \le \pi \\ 0 \le t \le 2\pi \end{array} \right\}$$

(d)

$$\mathbf{X}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix} = \begin{pmatrix} (a + b\cos t)\cos s \\ (a + b\cos t)\sin s \\ b\sin t \end{pmatrix}, \qquad D = \left\{ \begin{array}{c} 0 \le s \le 2\pi \\ 0 \le t \le 2\pi \end{array} \right\}$$

(e, kind of an extra bonus stumper)

$$\mathbf{X}(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{pmatrix} = \begin{pmatrix} \left(1 + t\cos\frac{s}{2}\right)\cos s \\ \left(1 + t\cos\frac{s}{2}\right)\sin s \\ t\sin\frac{s}{2} \end{pmatrix}, \qquad D = \left\{ \begin{array}{c} 0 \le s \le 2\pi \\ -\frac{1}{2} \le t \le \frac{1}{2} \end{array} \right\}$$

2. (Colley problem 7.1.23) Find the area of the surface cut from the paraboloid $z = 2x^2 + 2y^2$ by the planes $z = 2$ and $z = 8$.

3. (Colley problem 7.1.29) Consider a surface $\mathcal{S}$ defined in spherical coordinates by the equation $\rho = f(\varphi, \theta)$ where (φ, θ) varies through a region D in the $\varphi\theta$ plane and $f(\varphi, \theta)$ is non-negative. Compute the surface area of the surface $\mathcal{S}$.