# 18.022 Practice Problems, 11/20/2013

*Recitation Instructor: Homer Reid*

**1.** (Colley problem 6.2.10) Consider the cycloid curve defined by the equations

$$x = a(t - \sin t), \qquad y = a(1 - \cos t)$$

where $a$ is a positive constant.

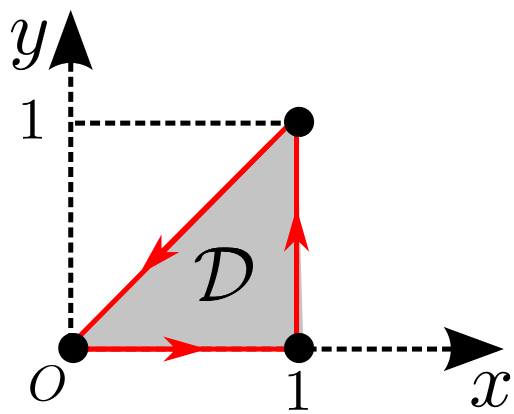
**(a)** Write a one-dimensional integral (a line-integral) expression for the area under one arch of this curve.

**(b)** Write a *two-dimensional* integral expression for the area under one arch of this curve.

**2.** A common problem in modern computational electromagnetism is to compute the electrostatic potential and field due to a constant surface charge density confined to a region $\mathcal{D}$ of the $xy$ plane. If the constant surface charge density throughout the region $D$ is $\sigma$, then the potential at a point $\mathbf{x}$ (in appropriate units) is

$$\phi(\mathbf{x}) = \sigma \iint_{\mathcal{D}} \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}. \tag{1}$$

Now suppose $\mathcal{D}$ is the triangle shown below.




**(a)** Write down a two-dimensional integral expression for the electrostatic potential $\phi(\mathbf{x})$ due to this triangle at the point $\mathbf{x} = O$ (the origin). Work in units such that $\sigma = 1$.

**(b)** Now write down a *one-dimensional* integral expression for the same quantity. *Hint, part 1:* The solution involves the two-dimensional vector field $\mathbf{F}(x, y)$ defined by[1]

$$\mathbf{F}(x, y) = -\frac{y}{\sqrt{x^2 + y^2}}\hat{\mathbf{x}} + \frac{x}{\sqrt{x^2 + y^2}}\hat{\mathbf{y}}. \tag{2}$$

**(c)** Evaluate the integrals you wrote down in parts (a) and (b) to compute the electrostatic potential at the origin. Note: For the two-dimensional case, it is particularly convenient—though not strictly necessary—to change variables to $u(x, y) = x, v(x, y) = y/x$. (This type of variable transformation is called a *Duffy transformation.*[2])

**Extra bonus stumper.** Suppose that instead of the potential at the origin $O$, we want to find the potential at some arbitrary point $\mathbf{x}$ (not necessarily lying in the $xy$ plane). Generalize the preceding treatment to handle this case.

[1] *Hint, part 2:* Although it is not necessary to solve this problem, you may benefit from the knowledge that, if we work in cylindrical coordinates, the expression for the curl of a vector-valued function $\mathbf{F}(r, \theta, z)$ is

$$\nabla \times \mathbf{F} = \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \frac{1}{r}\left(\frac{\partial (rF_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta}\right)\hat{\boldsymbol{z}}.$$

[2] Reference: M. D. Duffy, "Quadrature Over a Pyramid or Cube of Integrands with a Singularity at a Vertex," *SIAM Journal on Numerical Analysis* **19** 1260-1262 (1982).

**Extra extra bonus bonus stumper stumper.** The case we considered in this problem was for electro*statics*, i.e. electromagnetism at zero frequency, in which case the potential due to a point source at distance $r$ is (in appropriate units) $1/r$. In electro*dynamics* at frequency $\omega \neq 0$, the potential of a point source at a distance $r$ is modified according to

$$\frac{1}{r} \qquad \Longrightarrow \qquad \frac{e^{ikr}}{r}$$

where $k = \omega/c$ is proportional to the frequency ($c$ is the speed of light). Equation (1) is modified to read

$$\phi(\mathbf{x}) = \sigma \iint_{\mathcal{D}} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}'.$$

Repeat part **(b)** above for this case – that is, find a nonzero-frequency analogue of the function $\mathbf{F}(x, y)$ in equation (2). (The final answer is simpler than you might expect.)

**3.** (Colley problem 6.3.14) One of the following two vector fields is conservative, and the other is not. Determine which is which, and find a potential function for the conservative field.

**(a)** $\mathbf{F}(x, y, z) = xy^2z^3\,\mathbf{i} + 2x^2y\,\mathbf{j} + 3x^2y^2z^2\,\mathbf{k}$.

**(b)** $\mathbf{G}(x, y, z) = 2xy\,\mathbf{i} + (x^2 + 2yz)\,\mathbf{j} + y^2\,\mathbf{k}$.