18.022 Practice Problems, 11/18/2013

Recitation Instructor: Homer Reid

1. (Colley problem 6.1.20) Let $\mathcal{C}$ be the curve defined by the equation $y^2 = x^3$.

(a) Consider a piece of wire bent into the shape of this curve and running from the point $(1,-1)$ to the point $(1,1)$. Suppose the mass density (the mass per unit length) of the wire is $\lambda(x) = \frac{1}{\sqrt{x}}$. Compute the **(i)** length, **(ii)** mass of the wire segment.

(b) Suppose the electric field in the vicinity of the wire is given by $\mathbf{E}(\mathbf{x}) = x^2 y\,\hat{\mathbf{x}} - xy\,\hat{\mathbf{y}}$. Compute the work required to transport a unit-strength electric charge along the wire segment from $(1,-1)$ to $(1,1)$.

2.

(a) Use Green's theorem to compute the area of the rectangle lying in the xy plane with corners at the four points

$$(0,0), \quad (a,0), \quad (a,b), \quad (0,b)$$

where a, b are positive constants.

(b) (Colley problem 6.2.10) Use Green's theorem to compute the area under one arch of the cycloid

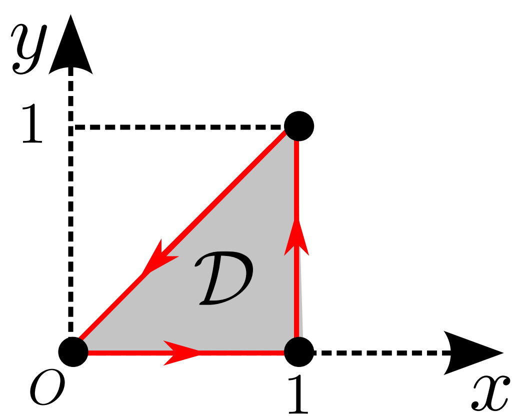
$$x = a(t - \sin t), \qquad y = a(1 - \cos t)$$

where a is a positive constant.

3. A common problem in modern computational electromagnetism is to compute the electrostatic potential and field due to a constant surface charge density confined to a region $\mathcal{D}$ of the xy plane. If the constant surface charge density throughout the region $\mathcal{D}$ is σ, then the potential at a point $\mathbf{x}$ (in appropriate units) is

$$\phi(\mathbf{x}) = \sigma \iint_{\mathcal{D}} \frac{d\mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|}. \tag{1}$$

Now suppose $\mathcal{D}$ is the triangle shown below.



(a) Write down a two-dimensional integral expression for the electrostatic potential $\phi(\mathbf{x})$ due to this triangle at the point $\mathbf{x} = O$ (the origin). Work in units such that $\sigma = 1$.

(b) Now write down a *one-dimensional* integral expression for the same quantity. Hint: The solution involves the two-dimensional vector field $\mathbf{F}(x,y)$ defined by

$$\mathbf{F}(x,y) = -\frac{y}{\sqrt{x^2+y^2}}\hat{\mathbf{x}} + \frac{x}{\sqrt{x^2+y^2}}\hat{\mathbf{y}}. \tag{2}$$

(c) Evaluate the integrals you wrote down in parts (a) and (b) to compute the electrostatic potential at the origin.

Extra bonus stumper. The case we considered in this problem was for electro*statics*, i.e. electromagnetism at zero frequency. In electro*dynamics* at frequency $\omega \neq 0$, the potential of a point source at a distance r is modified according to

$$\frac{1}{r} \qquad \Longrightarrow \qquad \frac{e^{ikr}}{r}$$

where $k = \omega/c$ is proportional to the frequency (c is the speed of light). Equation (1) is modified to read

$$\phi(\mathbf{x}) = \sigma \iint_{\mathcal{D}} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}'.$$

Repeat part **(b)** for this case – that is, find a nonzero-frequency analogue of the function $\mathbf{F}(x, y)$ in equation (2). (The final answer is simpler than you might expect.)