18.022 Practice Problems, 11/06/2013

Recitation Instructor: Homer Reid

1. (Colley problem 5.2.13, the last problem from Monday's worksheet). Integrate the function $f(x,y) = (x+y)$ over the region bounded by $x+y=2$ and $y^2-2y-x=0$.

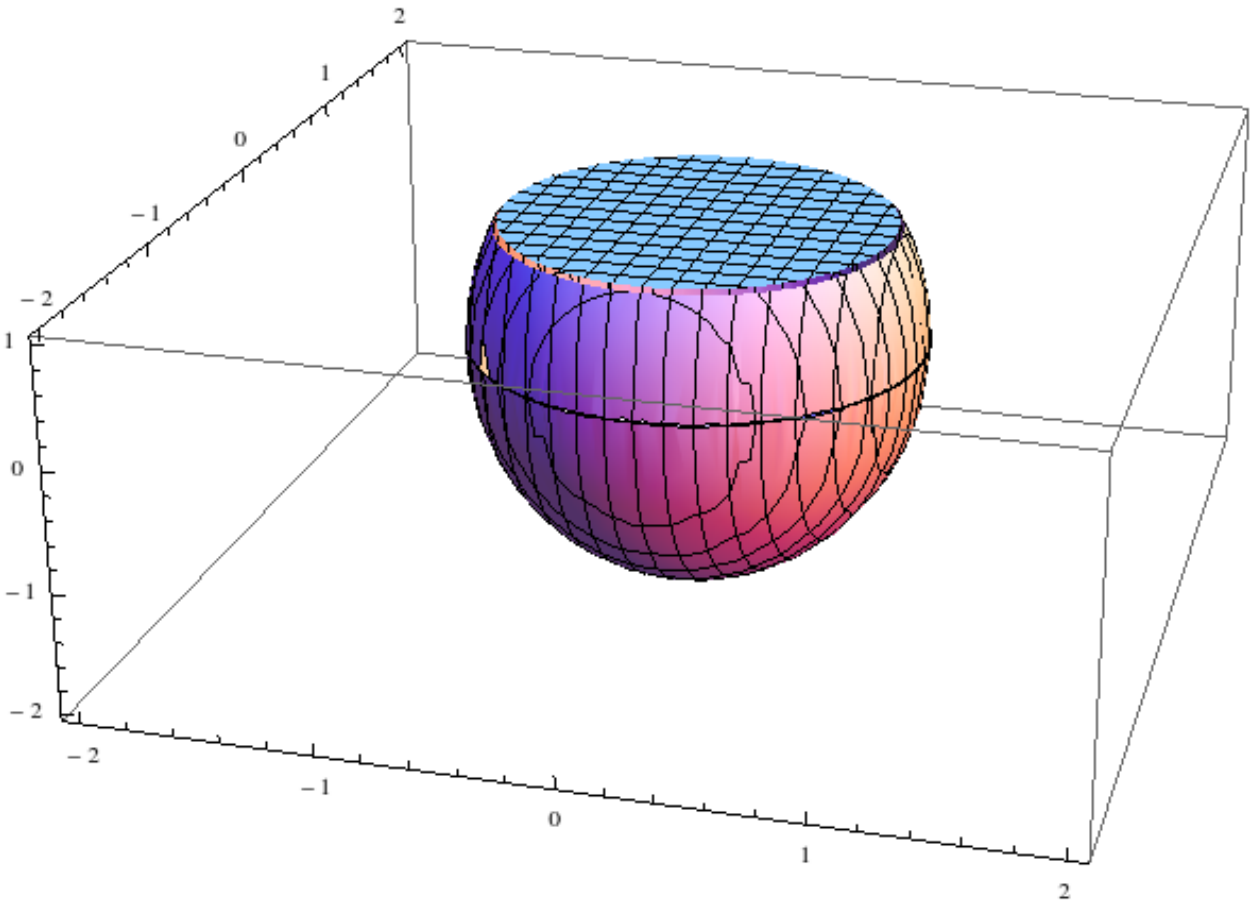
2. (Colley problem 5.3.13) Consider the following sum of two double integrals. Rewrite each integral with the order of integration reversed, then express the sum of the two integrals as a single double integral.

$$\int_0^8 \int_0^{\sqrt{y/3}} y\,dx\,dy \;+\; \int_8^{12} \int_{\sqrt{y-8}}^{\sqrt{y/3}} y\,dx\,dy$$

3. While leading an archaeology expedition you discover a massive dinosaur egg of ellipsoidal shape. The original volume occupied by the egg may be described as the set of points (x,y,z) satisfying

$$\left(\frac{x}{L}\right)^2 + \left(\frac{y}{L}\right)^2 + \left(\frac{z}{H}\right)^2 = 1$$

where $L=1, H=2$. However, in attempting to extract the egg from the ground, one of your dozy[1] assistants accidentally shears off a portion of the egg with his shovel, so that now the remaining volume of the egg is bounded on top by the intersection between the original ellipsoid and the plane $z=1$.



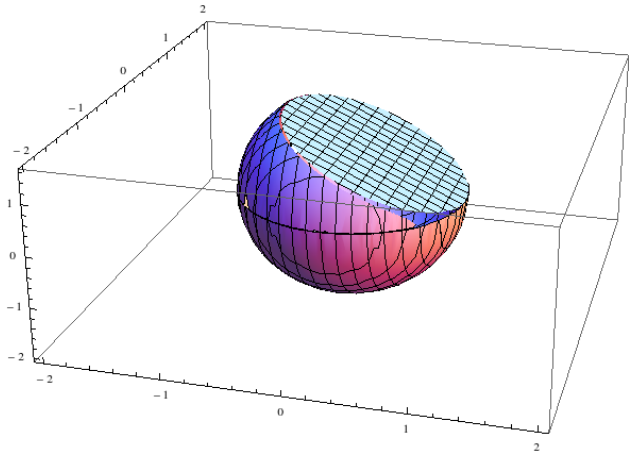
[1] Britishisms FTW!

To design a harness for airlifting the egg away from the dig site by helicopter, you need to know **(a)** the volume of the egg, **(b)** the center of mass of the egg. The *center of mass* of a volume V of constant mass density is the point $\mathbf{X}_0 = (X_0, Y_0, Z_0)$ whose coordinates are given by

$$X_0 = \int\int\int_V x\,dx\,dy\,dx, \qquad Y_0 = \int\int\int_V y\,dx\,dy\,dx, \qquad Z_0 = \int\int\int_V z\,dx\,dy\,dx.$$

Compute the volume and the center of mass of the sheared egg.

Extra bonus stumper. Suppose the dozy assistant shears off a slanted portion of the egg, so that the remaining volume of the sheared egg is bounded on top by the intersection of the original ellipsoid and the plane $x + z = 1$. Re-solve the above problem in this case.



4. Consider the following integral:

$$\int_0^1 \int_0^x f(x+y)g(x-y)dy\,dx.$$

where f and g are unspecified functions. Rewrite this integral under the change of variables $u = x + y, v = x - y$.

5.(a) Evaluate the two-dimensional integral

$$\mathcal{I}_2(\alpha) \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)}dx\,dy.$$

(b) Evaluate the one-dimensional integral

$$\mathcal{I}_1(\alpha) \equiv \int_{-\infty}^{\infty} e^{-\alpha x^2}dx.$$

(c) Evaluate the following one-dimensional integrals:

$$\int_{-\infty}^{\infty} xe^{-\alpha x^2}dx, \qquad \int_{-\infty}^{\infty} x^2e^{-\alpha x^2}dx.$$