4. (20pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth curve parametrised by arclength. Let $a \in I$ and suppose that

$$\vec{T}(a) = \frac{4}{9}\hat{\imath} - \frac{7}{9}\hat{\imath} - \frac{4}{9}\hat{k}, \quad \vec{B}(a) = \frac{1}{9}\hat{\imath} - \frac{4}{9}\hat{\imath} + \frac{8}{9}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = \hat{\imath} + 2\hat{\jmath}.$$

Find:

(i) the unit normal vector $\vec{N}(a)$.

(ii) the curvature $\kappa(a)$.

(iii) the torsion $\tau(a)$.

5. (20pts) Let $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the vector field given by $\vec{F}(x,y) = y\hat{\imath} + x\hat{\jmath}$.

(i) Is $\vec{F}$ a gradient field (that is, is $\vec{F}$ conservative)? Why?

(ii) Is $\vec{F}$ incompressible?

(iii) Find a flow line that passes through the point $(1,0)$.

(iv) Find a flow line that passes through the point (a,b), where $a^2 > b^2$.

2. (20pts) Suppose that $F\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is differentiable at $P = (-1,4)$ with derivative

$$DF(-1,4) = \begin{pmatrix} -1 & 1 \\ 3 & -2 \\ -2 & -1 \end{pmatrix}.$$

Suppose that $F(-1,4) = (1,-1,3)$. Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be the function $f(x,y) = \|F(x,y)\|$.

(i) Show that the function $f(x,y)$ is differentiable at P.

(ii) Find $Df(-1,4)$.

3. (20pts) Let

$$S = \{\, (x,y,z) \in \mathbb{R}^3 \mid x^3y + y^2z^3 + zx^2 = 3 \,\}.$$

(i) Show that S is the graph of a function $z = f(x,y)$ in a neighbourhood of $P = (1,-2,1)$.

(ii) Find the partial derivatives

$$\frac{\partial f}{\partial x}(1,-2) \quad \text{and} \quad \frac{\partial f}{\partial y}(1,-2).$$

4. (20pts) Let $\vec{r}\colon I \longrightarrow \mathbb{R}^3$ be a regular smooth curve parametrised by arclength. Let $a \in I$ and suppose that

$$\vec{N}(a) = \frac{2}{7}\hat{\imath} - \frac{6}{7}\hat{\jmath} - \frac{3}{7}\hat{k}, \quad \vec{B}(a) = \frac{3}{7}\hat{\imath} - \frac{2}{7}\hat{\jmath} + \frac{6}{7}\hat{k}, \quad \frac{d\vec{N}}{ds}(a) = \cancel{-12\hat{\imath} + \hat{\jmath} - 10\hat{k}}\ 12\hat{\imath} - \hat{\jmath} + 10\hat{k}.$$

Find:

(i) $\vec{T}(a)$.

(ii) $\kappa(a)$

(iii) $\tau(a)$

5. (20pts) Let $\vec{F}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the vector field given by $f(x,y) = y\hat{\imath} - 2\hat{\jmath}$.

(i) Is $\vec{F}$ a gradient field (that is, is $\vec{F}$ conservative)? Why?

(ii) Is $\vec{F}$ incompressible?

(iii) Find a flow line that passes through the point (a,b).

Problem 1. (20 points)

Let $f\colon \mathbf{R}^2 \to \mathbf{R}$ be defined by $f(x,y) = x^2y^2 - x$.

a) Find the gradient vector of f at $(1,2)$.

b) Find an equation for the tangent plane to the graph of f at $(1,2,3)$.

c) Use a linear approximation to estimate the value of $f(1.1, 1.9)$.

d) Find the directional derivative of f at $(1,2)$ in the direction of $\mathbf{v} = \frac{4}{5}\mathbf{i} - \frac{3}{5}\mathbf{j}$.

Problem 2. (20 points)

Let $\mathbf{F}\colon \mathbf{R}^5 \to \mathbf{R}^2$ be of class C^1. Suppose that

$$D\mathbf{F}(2,-1,1,4,1) = \begin{bmatrix} 4 & 0 & 1 & 2 & -1 \\ 1 & 0 & 0 & 1 & 2 \end{bmatrix}$$

a) Show that there is an open subset $U \subset \mathbf{R}^3$ containing $(2,-1,1)$ such that, for all $(x_1,x_2,x_3) \in U$, the system of equations

$$\mathbf{F}(x_1,x_2,x_3,x_4,x_5) = \mathbf{F}(2,-1,1,4,1)$$

has a unique solution $(x_4,x_5) = \mathbf{f}(x_1,x_2,x_3)$. Show that $\mathbf{f}\colon U \to \mathbf{R}^2$ is of class C^1.
b) Find $D\mathbf{f}(2,-1,1)$.

Problem 3. (20 points)

Let $(x(r,\theta), y(r,\theta))$ be the cartesian coordinates corresponding to the polar coordinates (r,θ). Suppose that $f(x,y)$ is differentiable at $(4,-3)$ with

$$\frac{\partial f}{\partial x}(4,-3) = 2, \qquad \frac{\partial f}{\partial y}(4,-3) = -1,$$

and let $g(r,\theta) = f(x(r,\theta), y(r,\theta))$. Find the partial derivatives $\partial g/\partial r(c,\alpha)$ and $\partial g/\partial\theta(c,\alpha)$ at a point (c,α) such that $(x(c,\alpha), y(c,\alpha)) = (4,-3)$.

Problem 5. (20 points)

Let $\mathbf{F}\colon \mathbf{R}^2 \to \mathbf{R}^2$ be the vector field defined by $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$.

a) Show that $\mathbf{F}$ is a gradient field.

b) Find the flow line for $\mathbf{F}$ that passes through the point $(a,b) \neq (0,0)$ at time $t = 0$.

Problem 4. (20 points)

Let $\mathbf{r}\colon I \to \mathbf{R}^3$ be a regular smooth curve parametrized by arclength. Let $a \in I$ and suppose that

$$\mathbf{T}(a) = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}, \qquad \frac{d\mathbf{T}}{ds}(a) = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k},$$
$$\mathbf{N}(a) = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}, \qquad \frac{d\mathbf{N}}{ds}(a) = -5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

Find each of the following:

a) The binormal vector $\mathbf{B}(a)$.

b) The curvature $\kappa(a)$.

c) The torsion $\tau(a)$.

2. (20pts) Suppose that $F\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ is differentiable at $P = (3,-2,1)$ with derivative

$$DF(3,-2,1) = \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & -3 \end{pmatrix}.$$

Suppose that $F(3,-2,1) = (1,-3)$. Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be the function $f(x,y,z) = \|F(x,y,z)\|$.
(i) Show that the function $f(x,y,z)$ is differentiable at P.

3. (20pts) Let $F\colon \mathbb{R}^4 \longrightarrow \mathbb{R}^2$ be a C^1 function. Suppose that

$$DF(3,1,0,-1) = \begin{pmatrix} 1 & 3 & 1 & 3 \\ -1 & 2 & -1 & -2 \end{pmatrix}.$$

(a) Show that there is an open subset $U \subset \mathbb{R}^2$ containing $(3,1)$ and an open subset $V \subset \mathbb{R}^2$ containing $(0,-1)$ such that for all $(x,y) \in U$, the system of equations

$$F(x,y,z,w) = F(3,1,0,-1),$$

has the unique solution

$$(z,w) = (f_1(x,y), f_2(x,y)) \quad \text{with} \quad (z,w) \in V.$$