# 18.022 Practice Problems, 10/21/2013

*Recitation Instructor: Homer Reid*

**1.** Consider a cyclist riding a bicycle with wheels of radius $R = \frac{1}{\pi} \approx 0.32$ meters. Suppose that, starting at time $t = 1$, the cyclist travels in the $xy$ plane in such a way that her coordinates as a function of time are

$$x(t) = v_0 t, \qquad y(t) = \frac{2}{3}(v_0 t - 1)^{\frac{3}{2}}$$

where $v_0$ is a constant velocity.[1]

**(a)** First suppose $v_0 = 1$ m/s. Compute the $x, y$ coordinates of the cyclist at time $t = 10$ seconds. Find the (two-dimensional) velocity vector $\mathbf{v}(t)$ and the unit tangent vector $\mathbf{T}(t)$.

**(b)** Again supposing $v_0 = 1$ m/s, compute the $x, y$ coordinates of the cyclist after her wheels have completed 9 complete rotations.

**(c)** Now suppose $v_0 = 2$ m/s. Repeat parts **(a)** and **(b)**.

**(d)** Reparameterize the cyclist's trajectory in terms of arc length. That is, obtain functions $x(s), y(s)$ that give the cyclist's coordinates as a function of the total distance she has traveled. Compute the unit tangent vector $\mathbf{T}(s)$ as a function of arc length. Compute the curvature $\kappa(s)$.

**(e)** Now thinking of the trajectory as a *three-dimensional* curve, compute the three-dimensional moving frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and the torsion $\tau$.

**2.** Now suppose that the cyclist in the previous problem switches to a bicycle equipped with jet thrusters that allow traversal of three-dimensional curves. Suppose that, starting at time $t = 1$, the cyclist executes the following trajectory:

$$x(t) = at, \qquad y(t) = bt^2, \qquad z(t) = ct^3$$

where $a, b, c > 0$ are constants with units of velocity, velocity/time, and velocity/(time)$^2$ respectively.

**(a)** Write an expression for $s(t)$, the total distance (arc length) the cyclist has traveled at time $t$.

**(b)** Compute the moving frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, the curvature $\kappa$, and the torsion $\tau$.

[1]Note: The units do not quite make sense here. The way to get them right is to put

$$y(t) = \frac{2}{3}\left(\frac{v_0 t}{y_0} - 1\right)^{\frac{3}{2}}$$

where $y_0$ is a parameter with units of length. The equations given above correspond to the case $y_0 = 1$ meter. This detail is not important for working out the mathematics of this example.

**3.** On the planet Krypton, persons convicted of high crimes are sentenced to a term in the *Phantom Zone*, a two-dimensional planar prison.

General Zod (together with his goons Ursa and Non) has recently been sentenced to a term in the Phantom Zone. While in prison, the General would like to keep up his favorite hobby of riding around on a tricycle equipped with jet thrusters which allow it to traverse the trajectories of various three-dimensional space curves. The General would like to know which of his favorite trajectories will be possible to execute inside the Phantom Zone.

For which of the following space curves is it possible to find a rotation of coordinates that allows the entire curve to be confined within a plane?

**(a)**

$$\mathbf{x}(t) = (e^t \cos t, e^t \sin t, e^t).$$

**(b)**

$$\mathbf{x}(t) = (e^t \cos\theta \cos t, e^t \sin t, -e^t \sin\theta \cos t).$$

(where $\theta$ is a constant.)