On The Number Of Primes Below A Given Magnitude

The Riemann Zeta Function And How To Use It

Homer Reid

CP Group Talk

1/17/2008

First Definition of the Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}$$
$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

First Definition of the Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \qquad \zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

First Definition of the Zeta Function

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}$$
$$= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots$$

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \qquad \zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

This definition only works for Re $s > 1$.

What Does $\zeta(s)$ Have To Do With Primes?

The Euler Product Formula

$$\prod_{p \text{ prime}} \frac{1}{\left(1 - \frac{1}{p^s}\right)} = \left(\frac{1}{1 - \frac{1}{2^s}}\right)\left(\frac{1}{1 - \frac{1}{3^s}}\right)\left(\frac{1}{1 - \frac{1}{5^s}}\right)\cdots$$

What Does $\zeta(s)$ Have To Do With Primes?

The Euler Product Formula

$$\prod_{p \text{ prime}} \frac{1}{\left(1-\frac{1}{p^s}\right)} = \left(\frac{1}{1-\frac{1}{2^s}}\right)\left(\frac{1}{1-\frac{1}{3^s}}\right)\left(\frac{1}{1-\frac{1}{5^s}}\right)\cdots$$

$$= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\cdots\right)\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\cdots\right)\left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\cdots\right)\cdots$$

$$\prod_{p \text{ prime}} \frac{1}{\left(1-\frac{1}{p^s}\right)} = \left(\frac{1}{1-\frac{1}{2^s}}\right)\left(\frac{1}{1-\frac{1}{3^s}}\right)\left(\frac{1}{1-\frac{1}{5^s}}\right)\cdots$$

$$= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\cdots\right)\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\cdots\right)\left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\cdots\right)\cdots$$

$$= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{2^{2s}}+\frac{1}{5^s}+\frac{1}{2^s3^s}+\frac{1}{7^s}+\frac{1}{2^{3s}}+\frac{1}{3^{2s}}+\frac{1}{2^s5^s}+\cdots$$

What Does $\zeta(s)$ Have To Do With Primes?

The Euler Product Formula

$$\prod_{p \text{ prime}} \frac{1}{\left(1-\frac{1}{p^s}\right)} = \left(\frac{1}{1-\frac{1}{2^s}}\right)\left(\frac{1}{1-\frac{1}{3^s}}\right)\left(\frac{1}{1-\frac{1}{5^s}}\right)\cdots$$

$$= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\cdots\right)\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\cdots\right)\left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\cdots\right)\cdots$$

$$= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{2^{2s}}+\frac{1}{5^s}+\frac{1}{2^s3^s}+\frac{1}{7^s}+\frac{1}{2^{3s}}+\frac{1}{3^{2s}}+\frac{1}{2^s5^s}+\cdots$$

$$= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\frac{1}{9^s}+\frac{1}{10^s}+\cdots \quad \left(\begin{array}{l}\text{fundamental}\\ \text{theorem of}\\ \text{arithmetic}\end{array}\right)$$

What Does $\zeta(s)$ Have To Do With Primes?

The Euler Product Formula

$$\prod_{p \text{ prime}} \frac{1}{\left(1-\frac{1}{p^s}\right)} = \left(\frac{1}{1-\frac{1}{2^s}}\right)\left(\frac{1}{1-\frac{1}{3^s}}\right)\left(\frac{1}{1-\frac{1}{5^s}}\right)\cdots$$

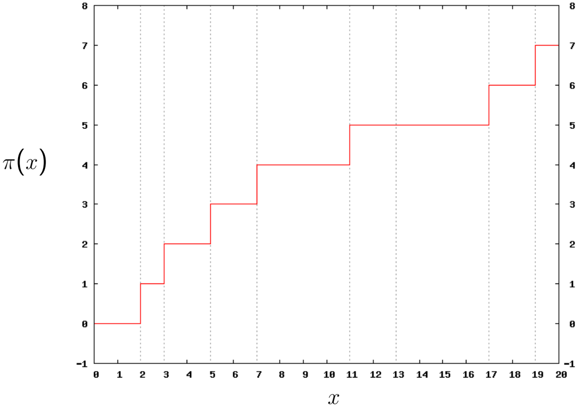
$$= \left(1+\frac{1}{2^s}+\frac{1}{2^{2s}}+\cdots\right)\left(1+\frac{1}{3^s}+\frac{1}{3^{2s}}+\cdots\right)\left(1+\frac{1}{5^s}+\frac{1}{5^{2s}}+\cdots\right)\cdots$$

$$= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{2^{2s}}+\frac{1}{5^s}+\frac{1}{2^s3^s}+\frac{1}{7^s}+\frac{1}{2^{3s}}+\frac{1}{3^{2s}}+\frac{1}{2^s5^s}+\cdots$$

$$= 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\frac{1}{7^s}+\frac{1}{8^s}+\frac{1}{9^s}+\frac{1}{10^s}+\cdots \left(\begin{array}{l}\text{fundamental}\\ \text{theorem of}\\ \text{arithmetic}\end{array}\right)$$

$$= \zeta(s).$$

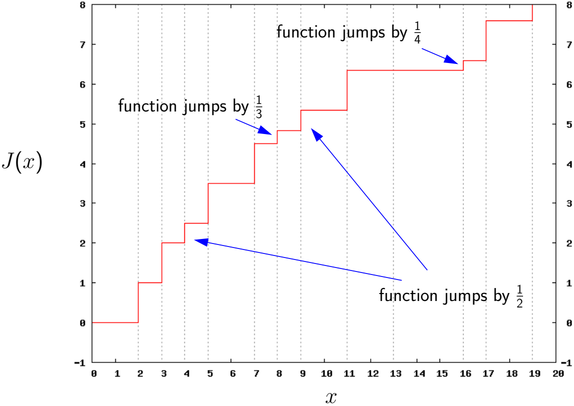
$$\pi(x) \equiv \text{number of primes} \leq x$$



What Does $\zeta(s)$ Have To Do With Primes?

The Function $J(x)$

$$J(x) = (\text{number of primes} \le x) + \frac{1}{2}(\text{number of prime squares} \le x) + \frac{1}{3}(\text{number of prime cubes} \le x) + \cdots$$
$$= \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots$$



What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$

Recall the Euler product formula:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \Longrightarrow \qquad \log \zeta(s) = -\sum \log\left(1 - \frac{1}{p^s}\right)$$

What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$

Recall the Euler product formula:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \Longrightarrow \qquad \log\zeta(s) = -\sum \log\left(1 - \frac{1}{p^s}\right)$$

Expand $\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$

What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$

Recall the Euler product formula:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \Longrightarrow \qquad \log \zeta(s) = -\sum \log\left(1 - \frac{1}{p^s}\right)$$

Expand $\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$

$$\begin{aligned}\log \zeta(s) = \quad & \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \cdots \\ & + \frac{1}{2}\left(\frac{1}{2^{2s}} + \frac{1}{3^{2s}} + \frac{1}{5^{2s}} + \cdots\right) \\ & + \frac{1}{3}\left(\frac{1}{2^{3s}} + \frac{1}{3^{3s}} + \frac{1}{5^{3s}} + \cdots\right)\end{aligned}$$

What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$

Recall the Euler product formula:

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad \Longrightarrow \qquad \log\zeta(s) = -\sum \log\left(1 - \frac{1}{p^s}\right)$$

Expand $\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$

$$\begin{aligned}\log\zeta(s) = \quad & \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \cdots \\ & + \frac{1}{2}\left(\frac{1}{2^{2s}} + \frac{1}{3^{2s}} + \frac{1}{5^{2s}} + \cdots\right) \\ & + \frac{1}{3}\left(\frac{1}{2^{3s}} + \frac{1}{3^{3s}} + \frac{1}{5^{3s}} + \cdots\right) \\ = & \sum_{\text{All integers } n \text{ at which } J(n) \text{ has a jump}} (\text{magnitude of jump in } J) \cdot \frac{1}{n^s}\end{aligned}$$

What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$, continued

$$\log \zeta(s) = \sum_{\text{All integers } n \text{ at which } J(n) \text{ has a jump}} (\text{magnitude of jump in } J) \cdot \frac{1}{n^s}$$

What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$, continued

$$\log \zeta(s) = \sum_{\text{All integers } n \text{ at which } J(n) \text{ has a jump}} (\text{magnitude of jump in } J) \cdot \frac{1}{n^s}$$

$$= \int_0^\infty \frac{1}{x^s} dJ(x) \qquad \textit{(Stieltjes integral)}$$

What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$, continued

$$\log\zeta(s) = \sum_{\text{All integers } n \text{ at which } J(n) \text{ has a jump}} (\text{magnitude of jump in } J)\cdot\frac{1}{n^s}$$

$$= \int_0^\infty \frac{1}{x^s}\,dJ(x) \qquad \textit{(Stieltjes integral)}$$

Integrate by parts:

$$= s\int_0^\infty \frac{1}{x^{s+1}} J(x)\,dx$$

What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$, continued

$$\log\zeta(s) = \sum_{\text{All integers } n \text{ at which } J(n) \text{ has a jump}} (\text{magnitude of jump in } J)\cdot\frac{1}{n^s}$$

$$= \int_0^\infty \frac{1}{x^s}\,dJ(x) \qquad \textit{(Stieltjes integral)}$$

Integrate by parts:

$$= s\int_0^\infty \frac{1}{x^{s+1}}J(x)\,dx$$

Now identify Mellin transform (Laplace transform with $e^{-st} \leftrightarrow x^{-s}$) pairs:

$$\frac{\log\zeta(s)}{s} = \int_0^\infty x^{-(s+1)}J(x)\,dx \quad\Longleftrightarrow\quad \boxed{J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{\log\zeta(s)}{s}x^s\,ds}$$

What Does $\zeta(s)$ Have To Do With Primes?

The connection between $J(x)$ and $\zeta(s)$, continued

$$\log\zeta(s) = \sum_{\text{All integers } n \text{ at which } J(n) \text{ has a jump}} (\text{magnitude of jump in } J)\cdot\frac{1}{n^s}$$

$$= \int_0^\infty \frac{1}{x^s} dJ(x) \qquad \textit{(Stieltjes integral)}$$

Integrate by parts:

$$= s\int_0^\infty \frac{1}{x^{s+1}} J(x)dx$$

Now identify Mellin transform (Laplace transform with $e^{-st} \leftrightarrow x^{-s}$) pairs:

$$\frac{\log\zeta(s)}{s} = \int_0^\infty x^{-(s+1)} J(x)dx \quad \Longleftrightarrow \quad \boxed{J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\log\zeta(s)}{s} x^s ds}$$

$\Longrightarrow$ Analytical formula for $J(x)$! But need $\zeta(s)$ at arbitrary complex s.

Second Definition of the Zeta Function

Extending ζ to the entire complex plane

- We need an analytic continuation of $\zeta(s) = \sum \frac{1}{n^s}$ to the entire complex s plane.

Second Definition of the Zeta Function

Extending ζ to the entire complex plane

- We need an analytic continuation of $\zeta(s) = \sum \frac{1}{n^s}$ to the entire complex s plane.
- Proceed in analogy to the Gamma function $\Pi(z)$, which is an analytic continuation of the factorial function $z!$ to the entire complex z plane:

$$\Pi(z) \quad \equiv \quad \int_0^\infty t^z e^{-t} dt \quad = \quad z! \text{ if } z \text{ integer, but also valid for all } z.$$

Second Definition of the Zeta Function

Extending ζ to the entire complex plane

- We need an analytic continuation of $\zeta(s) = \sum \frac{1}{n^s}$ to the entire complex s plane.
- Proceed in analogy to the Gamma function $\Pi(z)$, which is an analytic continuation of the factorial function $z!$ to the entire complex z plane:

$$\Pi(z) \quad \equiv \quad \int_0^\infty t^z e^{-t} dt \quad = \quad z! \text{ if } z \text{ integer, but also valid for all } z.$$

- To derive a similar formula for $\zeta(s)$, note

$$\int_0^\infty x^{s-1} e^{-nx} dx = \frac{\Pi(s-1)}{n^s}.$$

Second Definition of the Zeta Function

Extending ζ to the entire complex plane

- We need an analytic continuation of $\zeta(s) = \sum \frac{1}{n^s}$ to the entire complex s plane.
- Proceed in analogy to the Gamma function $\Pi(z)$, which is an analytic continuation of the factorial function $z!$ to the entire complex z plane:

$$\Pi(z) \quad \equiv \quad \int_0^\infty t^z e^{-t} dt \quad = \quad z! \text{ if } z \text{ integer, but also valid for all } z.$$

- To derive a similar formula for $\zeta(s)$, note

$$\int_0^\infty x^{s-1} e^{-nx} dx = \frac{\Pi(s-1)}{n^s}.$$

Sum both sides from $n = 1$ to ∞:

$$\int_0^\infty x^{s-1} \underbrace{\left(e^{-x} + e^{-2x} + \cdots\right)}_{=\frac{1}{e^x - 1}} \qquad = \qquad \Pi(s-1) \underbrace{\sum_{n=1}^\infty \frac{1}{n^s}}_{\zeta(s)}$$

Second Definition of the Zeta Function

Extending ζ to the entire complex plane

- We need an analytic continuation of $\zeta(s) = \sum \frac{1}{n^s}$ to the entire complex s plane.
- Proceed in analogy to the Gamma function $\Pi(z)$, which is an analytic continuation of the factorial function $z!$ to the entire complex z plane:

$$\Pi(z) \quad \equiv \quad \int_0^\infty t^z e^{-t} dt \quad = \quad z! \text{ if } z \text{ integer, but also valid for all } z.$$

- To derive a similar formula for $\zeta(s)$, note

$$\int_0^\infty x^{s-1} e^{-nx} dx = \frac{\Pi(s-1)}{n^s}.$$

Sum both sides from $n = 1$ to ∞:

$$\int_0^\infty x^{s-1} \underbrace{\left(e^{-x} + e^{-2x} + \cdots\right)}_{=\frac{1}{e^x - 1}} \qquad = \qquad \Pi(s-1) \underbrace{\sum_{n=1}^\infty \frac{1}{n^s}}_{\zeta(s)}$$

$$\Longrightarrow \qquad \zeta(s) = \frac{1}{\Pi(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Second Definition of the Zeta Function

Extending ζ to the entire complex plane, continued

$$\zeta(s) = \frac{1}{\Pi(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Second Definition of the Zeta Function

Extending ζ to the entire complex plane, continued

$$\zeta(s) = \frac{1}{\Pi(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Works for $s > 1$, but for $s \leq 1$ the integral diverges due to the blowup of the integrand near $x = 0$.

Second Definition of the Zeta Function

Extending ζ to the entire complex plane, continued

$$\zeta(s) = \frac{1}{\Pi(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Works for $s > 1$, but for $s \leq 1$ the integral diverges due to the blowup of the integrand near $x = 0$.

Riemann's brainstorm: Consider instead the contour integral $\int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$

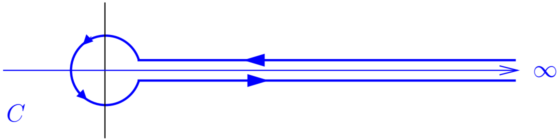
Second Definition of the Zeta Function

Extending ζ to the entire complex plane, continued

$$\zeta(s) = \frac{1}{\Pi(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Works for $s > 1$, but for $s \leq 1$ the integral diverges due to the blowup of the integrand near $x = 0$.

Riemann's brainstorm: Consider instead the contour integral $\int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$



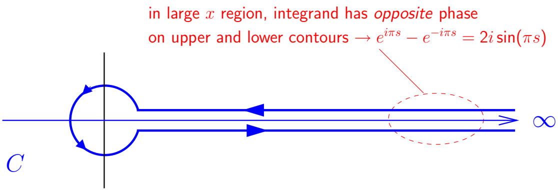
Second Definition of the Zeta Function

Extending ζ to the entire complex plane, continued

$$\zeta(s) = \frac{1}{\Pi(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Works for $s > 1$, but for $s \leq 1$ the integral diverges due to the blowup of the integrand near $x = 0$.

Riemann's brainstorm: Consider instead the contour integral $\displaystyle\int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$



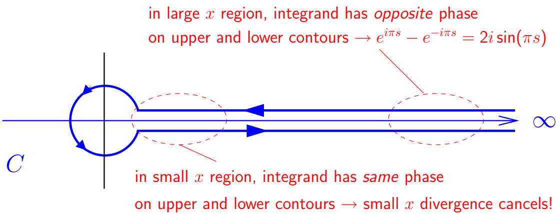
Second Definition of the Zeta Function

Extending ζ to the entire complex plane, continued

$$\zeta(s) = \frac{1}{\Pi(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Works for $s > 1$, but for $s \leq 1$ the integral diverges due to the blowup of the integrand near $x = 0$.

Riemann's brainstorm: Consider instead the contour integral $\displaystyle\int_C \frac{(-x)^s}{e^x - 1}\frac{dx}{x}$



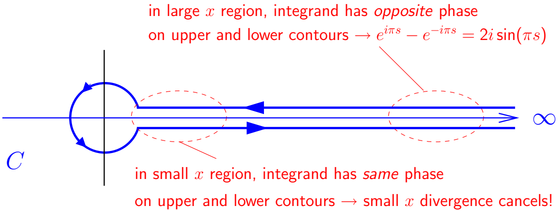
Second Definition of the Zeta Function

Extending ζ to the entire complex plane, continued

$$\zeta(s) = \frac{1}{\Pi(s-1)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx.$$

Works for $s > 1$, but for $s \le 1$ the integral diverges due to the blowup of the integrand near $x = 0$.

Riemann's brainstorm: Consider instead the contour integral $\int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$



$$\Longrightarrow \qquad \zeta(s) \equiv \frac{1}{2i \sin \pi s \Pi(s-1)} \int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = \frac{\Pi(-s)}{2\pi i} \int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$$

Fun Facts About $\zeta(s)$

Poles, zeroes, and the functional equation.

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x}$$

- $\zeta(s)$ has only one simple pole, at $s = 1$ with residue 1.

  $\Pi(-s)$ has poles at $s = 1, 2, 3, \cdots$, but the poles at $s = 2, 3, \cdots$ are cancelled by the behavior of the contour integral.

Fun Facts About $\zeta(s)$

Poles, zeroes, and the functional equation.

$$\boxed{\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_C \frac{(-x)^s}{e^x - 1} \frac{dx}{x}}$$

- $\zeta(s)$ has only one simple pole, at $s = 1$ with residue 1.

  $\Pi(-s)$ has poles at $s = 1, 2, 3, \cdots$, but the poles at $s = 2, 3, \cdots$ are cancelled by the behavior of the contour integral.

- $\zeta(s)$ satisfies a weird *functional equation* relating $\zeta(s)$ and $\zeta(1-s)$:

$$\zeta(s) = \Pi(-s)(2\pi)^{s-1} 2 \sin\left(\frac{s\pi}{2}\right) \zeta(1-s).$$

Fun Facts About $\zeta(s)$

Poles, zeroes, and the functional equation.

$$\boxed{\zeta(s) = \frac{\Pi(-s)}{2\pi i}\int_C \frac{(-x)^s}{e^x - 1}\frac{dx}{x}}$$

- $\zeta(s)$ has only one simple pole, at $s = 1$ with residue 1.

  $\Pi(-s)$ has poles at $s = 1, 2, 3, \cdots$, but the poles at $s = 2, 3, \cdots$ are cancelled by the behavior of the contour integral.

- $\zeta(s)$ satisfies a weird *functional equation* relating $\zeta(s)$ and $\zeta(1-s)$:

$$\zeta(s) = \Pi(-s)(2\pi)^{s-1} 2\sin\left(\frac{s\pi}{2}\right)\zeta(1-s).$$

- Combining the above two observations, it is convenient to define

$$\xi(s) \equiv \pi^{-s/2}\Pi\left(\frac{s}{2}\right)(s-1)\zeta(s)$$

in which case $\xi(s)$ is an *entire* function satisfying $\xi(s) = \xi(1-s)$.

Accounting explicitly for the pole at $s = 1$

Separate out the entire function $\xi(s)$ from $\zeta(s)$

$$\zeta(s) = \frac{1}{s-1} \cdot \frac{\pi^{s/2}}{\Pi\left(\frac{s}{2}\right)} \cdot \xi(s)$$

Since $\xi(s)$ is *entire*, we can write it as an "infinite degree polynomial:"

$$\xi(s) = \xi(0) \cdot \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

where the product is over all roots ρ of ξ (which are just the roots of ζ).

Accounting explicitly for the pole at $s = 1$

Separate out the entire function $\xi(s)$ from $\zeta(s)$

$$\zeta(s) = \frac{1}{s-1} \cdot \frac{\pi^{s/2}}{\Pi\left(\frac{s}{2}\right)} \cdot \xi(s)$$

Since $\xi(s)$ is *entire*, we can write it as an "infinite degree polynomial:"

$$\xi(s) = \xi(0) \cdot \prod_\rho \left(1 - \frac{s}{\rho}\right)$$

where the product is over all roots ρ of ξ (which are just the roots of ζ).

$$\Longrightarrow \qquad \log \zeta(s) = -\log(s-1) \; + \frac{s}{2}\log \pi - \log \Pi\left(\frac{s}{2}\right) + \log \xi(0) + \sum_\rho \log\left(1 - \frac{s}{\rho}\right).$$

Accounting explicitly for the pole at $s = 1$

Separate out the entire function $\xi(s)$ from $\zeta(s)$

$$\boxed{\zeta(s) = \frac{1}{s-1} \cdot \frac{\pi^{s/2}}{\Pi\left(\frac{s}{2}\right)} \cdot \xi(s)}$$

Since $\xi(s)$ is *entire*, we can write it as an "infinite degree polynomial:"

$$\xi(s) = \xi(0) \cdot \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

where the product is over all roots ρ of ξ (which are just the roots of ζ).

$$\Longrightarrow \qquad \log \zeta(s) = \boxed{-\log(s-1)} + \frac{s}{2}\log \pi - \log \Pi\left(\frac{s}{2}\right) + \log \xi(0) + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right).$$

Analytic Expression for $J(x)$

Recall from before: $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\log\zeta(s)}{s} x^s ds$

Analytic Expression for $J(x)$

Recall from before: $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\log\zeta(s)}{s} x^s ds$

Insert expression for $\log\zeta(s)$ from previous slide:

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[-\log(s-1) + \frac{s}{2}\log\pi - \log\Pi\left(\frac{s}{2}\right) + \log\xi(0) + \sum_\rho \log\left(1-\frac{s}{\rho}\right)\right] x^s \frac{ds}{s}$$

Analytic Expression for $J(x)$

Recall from before: $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\log\zeta(s)}{s} x^s ds$

Insert expression for $\log\zeta(s)$ from previous slide:

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left[-\log(s-1) + \frac{s}{2}\log\pi - \log\Pi\left(\frac{s}{2}\right) + \log\xi(0) + \sum_\rho \log\left(1-\frac{s}{\rho}\right)\right] x^s \frac{ds}{s}$$

$$= \quad \mathrm{Li}(x) \quad + \quad 0 \quad + \quad \int_x^\infty \frac{dt}{t(t^2-1)\log t} \quad + \quad \log\xi(0) \quad - \quad \sum_\rho \mathrm{Li}(x^\rho)$$

Analytic Expression for $J(x)$

Recall from before: $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\log\zeta(s)}{s} x^s ds$

Insert expression for $\log\zeta(s)$ from previous slide:

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[-\log(s-1) + \frac{s}{2}\log\pi - \log\Pi\left(\frac{s}{2}\right) + \log\xi(0) + \sum_\rho \log\left(1-\frac{s}{\rho}\right)\right] x^s \frac{ds}{s}$$

$$= \quad \mathrm{Li}(x) \quad + \quad 0 \quad + \quad \int_x^\infty \frac{dt}{t(t^2-1)\log t} \quad + \quad \log\xi(0) \quad - \quad \sum_\rho \mathrm{Li}(x^\rho)$$

where $\mathrm{Li}(x) \equiv \mathrm{P.V.}\int_0^x \frac{dt}{\log t}$ is the *logarithmic integral* function.

Analytic Expression for $J(x)$

Recall from before: $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\log\zeta(s)}{s} x^s ds$

Insert expression for $\log\zeta(s)$ from previous slide:

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left[-\log(s-1) + \frac{s}{2}\log\pi - \log\Pi\left(\frac{s}{2}\right) + \log\xi(0) + \sum_\rho \log\left(1-\frac{s}{\rho}\right)\right] x^s \frac{ds}{s}$$

$$= \quad \mathrm{Li}(x) \quad + \quad 0 \quad + \quad \int_x^\infty \frac{dt}{t(t^2-1)\log t} \quad + \quad \log\xi(0) \quad - \quad \sum_\rho \mathrm{Li}(x^\rho)$$

where $\mathrm{Li}(x) \equiv \mathrm{P.V.} \int_0^x \frac{dt}{\log t}$ is the *logarithmic integral* function.

It turns out that the first term in this expression for $J(x)$ dominates the other terms and is itself a good approximation to $J(x)$:

$$\boxed{J(x) \approx \mathrm{Li}(x) = \int_0^x \frac{dt}{\log t}.}$$

Analytic Expression for $\pi(x)$

So now we have an analytic expression for $J(x)$:

$$J(x) \approx \text{Li}(x).$$

Analytic Expression for $\pi(x)$

So now we have an analytic expression for $J(x)$:

$$J(x) \approx \mathrm{Li}(x).$$

But our *goal* was to get an expression for $\pi(x)$, which is related to $J(x)$ according to

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots .$$

Analytic Expression for $\pi(x)$

So now we have an analytic expression for $J(x)$:

$$J(x) \approx \mathrm{Li}(x).$$

But our *goal* was to get an expression for $\pi(x)$, which is related to $J(x)$ according to

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots .$$

Can we *invert* this relation to obtain a formula for $\pi(x)$ in terms of $J(x)$?

Begin by defining the *Möbius symbol.* Recall that any integer n has a unique prime decomposition:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}.$$

The Möbius Inversion Formula

Part 1: The Möbius Symbol

Begin by defining the *Möbius symbol.* Recall that any integer n has a unique prime decomposition:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}.$$

The *Möbius symbol* $\mu(n)$ characterizes the number and multiplicity of the prime factors of n :

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ contains any primes more than once} \\ +1 & \text{if } n = \text{product of even number of distinct prime factors} \\ -1 & \text{if } n = \text{product of odd of distinct prime factors} \end{cases}$$

The Möbius Inversion Formula

Part 1: The Möbius Symbol

Begin by defining the *Möbius symbol.* Recall that any integer n has a unique prime decomposition:

$$n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}.$$

The *Möbius symbol* $\mu(n)$ characterizes the number and multiplicity of the prime factors of n :

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ contains any primes more than once} \\ +1 & \text{if } n = \text{product of even number of distinct prime factors} \\ -1 & \text{if } n = \text{product of odd of distinct prime factors} \end{cases}$$

n	1	2	3	4	5	6	7	8	9	10	11
$\mu(n)$	1	-1	-1	0	-1	$+1$	-1	0	0	$+1$	-1

The *Möbius inversion formula*:

$$g(x) = \sum_{n=1}^{\infty} f(nx) \qquad \Longleftrightarrow \qquad f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx).$$

The *Möbius inversion formula*:

$$g(x)=\sum_{n=1}^{\infty} f(nx) \qquad \Longleftrightarrow \qquad f(x)=\sum_{n=1}^{\infty}\mu(n)g(nx).$$

$$\begin{array}{ccccccccccccccccc}
g(x) & = & f(x) & + & f(2x) & + & f(3x) & + & f(4x) & + & f(5x) & + & f(6x) & + & f(7x) & + & \cdots \\
-g(2x) & = & -f(2x) & - & f(4x) & - & f(6x) & - & f(8x) & - & f(10x) & - & f(12x) & - & f(14x) & - & \cdots \\
-g(3x) & = & -f(3x) & - & f(6x) & - & f(9x) & - & f(12x) & - & f(15x) & - & f(18x) & - & f(21x) & - & \cdots \\
-g(5x) & = & -f(5x) & - & f(10x) & - & f(15x) & - & f(20x) & - & f(25x) & - & f(30x) & - & f(35x) & - & \cdots \\
+g(6x) & = & f(6x) & + & f(12x) & = & f(18x) & + & f(24x) & + & f(30x) & + & f(36x) & + & f(42x) & + & \cdots
\end{array}$$

The *Möbius inversion formula*:

$$g(x) = \sum_{n=1}^{\infty} f(nx) \qquad \Longleftrightarrow \qquad f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx).$$

$$\begin{array}{rcccccccccccccccc}
g(x) & = & f(x) & + & \cancel{f(2x)} & + & f(3x) & + & f(4x) & + & f(5x) & + & f(6x) & + & f(7x) & + & \cdots \\
-g(2x) & = & -\cancel{f(2x)} & - & f(4x) & - & f(6x) & - & f(8x) & - & f(10x) & - & f(12x) & - & f(14x) & - & \cdots \\
-g(3x) & = & -f(3x) & - & f(6x) & - & f(9x) & - & f(12x) & - & f(15x) & - & f(18x) & - & f(21x) & - & \cdots \\
-g(5x) & = & -f(5x) & - & f(10x) & - & f(15x) & - & f(20x) & - & f(25x) & - & f(30x) & - & f(35x) & - & \cdots \\
+g(6x) & = & f(6x) & + & f(12x) & = & f(18x) & + & f(24x) & + & f(30x) & + & f(36x) & + & f(42x) & + & \cdots
\end{array}$$

The *Möbius inversion formula*:

$$g(x) = \sum_{n=1}^{\infty} f(nx) \qquad \Longleftrightarrow \qquad f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx).$$

$$\begin{array}{rcccccccccccccccc}
g(x) & = & f(x) & + & \cancel{f(2x)} & + & \cancel{f(3x)} & + & \cancel{f(4x)} & + & f(5x) & + & f(6x) & + & f(7x) & + & \cdots \\
-g(2x) & = & -\cancel{f(2x)} & - & \cancel{f(4x)} & - & f(6x) & - & f(8x) & - & f(10x) & - & f(12x) & - & f(14x) & - & \cdots \\
-g(3x) & = & -\cancel{f(3x)} & - & f(6x) & - & f(9x) & - & f(12x) & - & f(15x) & - & f(18x) & - & f(21x) & - & \cdots \\
-g(5x) & = & -f(5x) & - & f(10x) & - & f(15x) & - & f(20x) & - & f(25x) & - & f(30x) & - & f(35x) & - & \cdots \\
+g(6x) & = & f(6x) & + & f(12x) & = & f(18x) & + & f(24x) & + & f(30x) & + & f(36x) & + & f(42x) & + & \cdots
\end{array}$$

The Möbius Inversion Formula
Part 2: The Inversion Formula

The *Möbius inversion formula*:

$$g(x) = \sum_{n=1}^{\infty} f(nx) \qquad \Longleftrightarrow \qquad f(x) = \sum_{n=1}^{\infty} \mu(n) g(nx).$$

$$\begin{array}{rcccccccccccccccc}
g(x) & = & f(x) & + & \cancel{f(2x)} & + & \cancel{f(3x)} & + & \cancel{f(4x)} & + & \cancel{f(5x)} & + & \cancel{f(6x)} & + & \cancel{f(7x)} & + & \cdots \\
-g(2x) & = & -\cancel{f(2x)} & - & \cancel{f(4x)} & - & \cancel{f(6x)} & - & \cancel{f(8x)} & - & \cancel{f(10x)} & - & \cancel{f(12x)} & - & \cancel{f(14x)} & - & \cdots \\
-g(3x) & = & -\cancel{f(3x)} & - & \cancel{f(6x)} & - & \cancel{f(9x)} & - & \cancel{f(12x)} & - & \cancel{f(15x)} & - & \cancel{f(18x)} & - & \cancel{f(21x)} & - & \cdots \\
-g(5x) & = & -\cancel{f(5x)} & - & \cancel{f(10x)} & - & \cancel{f(15x)} & - & \cancel{f(20x)} & - & \cancel{f(25x)} & - & \cancel{f(30x)} & - & \cancel{f(35x)} & - & \cdots \\
+g(6x) & = & \cancel{f(6x)} & + & \cancel{f(12x)} & = & \cancel{f(18x)} & + & \cancel{f(24x)} & + & \cancel{f(30x)} & + & \cancel{f(36x)} & + & \cancel{f(42x)} & + & \cdots
\end{array}$$

Analytic Expression for $\pi(x)$

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots .$$

Analytic Expression for $\pi(x)$

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots .$$

Use Möbius inversion with $g(x) = J(x), \quad f(n) = \frac{1}{n}\pi(x^{1/n})$:

$$\boxed{\pi(x) = J(x) - \frac{1}{2}J(x^{1/2}) - \frac{1}{3}J(x^{1/3}) - \frac{1}{5}J(x^{1/5}) + \frac{1}{6}J(x^{1/6}) + \cdots .}$$

Analytic Expression for $\pi(x)$

$$J(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots .$$

Use Möbius inversion with $g(x) = J(x), \quad f(n) = \frac{1}{n}\pi(x^{1/n})$:

$$\boxed{\pi(x) = J(x) - \frac{1}{2}J(x^{1/2}) - \frac{1}{3}J(x^{1/3}) - \frac{1}{5}J(x^{1/5}) + \frac{1}{6}J(x^{1/6}) + \cdots .}$$

Since $J(x) \approx \text{Li}(x)$, the number of primes $\leq x$ is well approximated by

$$\boxed{\pi(x) = \text{Li}(x) - \frac{1}{2}\text{Li}(x^{1/2}) - \frac{1}{3}\text{Li}(x^{1/3}) - \frac{1}{5}\text{Li}(x^{1/5}) + \frac{1}{6}\text{Li}(x^{1/6}) + \cdots .}$$

Testing The Analytic Expression for $\pi(x)$

TABLE III[a]

x	Riemann's error	Gauss's error
1,000,000	30	130
2,000,000	−9	122
3,000,000	0	155
4,000,000	33	206
5,000,000	−64	125
6,000,000	24	228
7,000,000	−38	179
8,000,000	−6	223
9,000,000	−53	187
10,000,000	88	339

[a]From Lehmer [L9].

(Table taken from Edwards, *Riemann's Zeta Function.*)