Tapering Between Uniform and Periodic Dielectric Media in 1D

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August 24, 2006

Contents

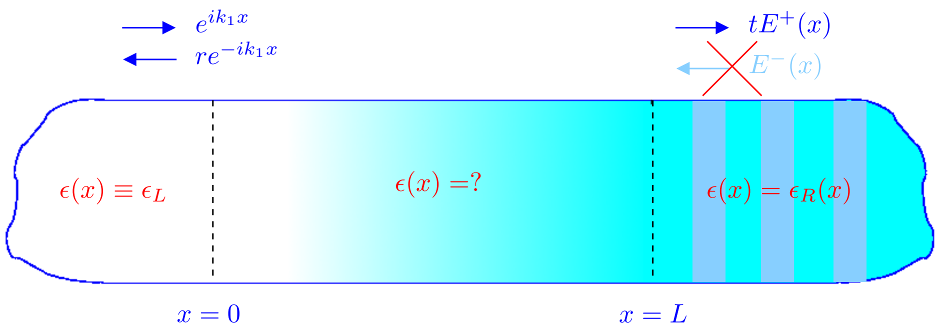

Figure 1: Cartoon depiction of 1D taper problem. The medium to the left of the taper has uniform dielectric constant ϵ_L. The medium to the right of the taper has spatially varying dielectric constant $\epsilon_R(x)$. In the left and right regions we can define left- and right- moving waves. We choose units such that the right-moving wave in the left region has amplitude 1; *the physics dictates that the left-moving wave in the right region has amplitude 0.* For a given dielectric function $\epsilon(x)$ in the taper region we can then solve a differential equation to determine the unknown constants r and t.

1 Overview

We consider the situation depicted in Figure 1. The medium on the left, which has uniform dielectric constant ϵ_L, is to be connected via a taper region to the region on the right, which has spatially varying dielectric constant $\epsilon_R(x)$. We imagine an electromagnetic wave of frequency ω entering the system from the left, and we seek to tune the dielectric function $\epsilon(x)$ in the taper region to miminize power reflected back into the source.

Although our previous discussions of this problem[1, 2] were fully general, we only provided explicit results for the simplest case of spatially *constant* $\epsilon_R(x)$. In this memo we illustrate how the same techniques may be used in cases where $\epsilon_R(x)$ is spatially *varying*.

We begin in Section 2 by considering the special case of a *linear* taper function. We demonstrate that this case may be solved *exactly*, for arbitrary periodic dielectric media, using the method outlined in Section 2 of [2]. We illustrate this computation for two particular types of periodic dielectric functions: a physically realistic step-function taper as cartooned in Figure 1, and a mathematically convenient sinusoidal taper.

In Section 3 we consider the case of an *arbitrary* taper function. We briefly review the integral equation method introduced in Section 3 of [2], in which the electric field for a given taper function may be computed to arbitrary accuracy, and we note that the method works just as well for periodic media as for uniform media. We illustrate the method by optimizing a taper between uniform and step-function periodic media.

2 Exact Solution for Linear Taper

2.1 Derivation of the Master Formula

In the region to the right of the taper (the periodic medium), each of the two transverse polarization modes of the electric field is determined by the equation

$$\left[\frac{d^2}{dx^2} + k^2(x)\right] E(x) = 0 \tag{1}$$

where

$$k^2(x) = \frac{\omega^2}{c^2}\epsilon_R(x).$$

In general, equation (1) is an ordinary second-order differential equation with two orthogonal solutions $f_1(x)$ and $f_2(x)$. Any linear combination of the form

$$E(x) = A_1 f_1(x) + A_2 f_2(x) \tag{2}$$

is thus a possible electric field; *which* linear combination actually obtains is dictated by the particular physics of the situation. For example, if we had a resonant cavity with walls at $x = x_1$ and $x = x_2$, the physics would dictate that $E(x_1) = E(x_2) = 0$, which, using (2), translates into a simple 2×2 linear system that we solve for A_1 and A_2.

On the other hand, the geometry of Figure 1 contains no such walls. Instead, the physical principle that dictates the choice of constants in (1) is that there be no electromagnetic power incident from the right. How do we translate this into a constraint on A_1 and A_2? Well, the physical requirement of no left-moving wave translates into the mathematical requirement that the time-average Poynting vector point rightward, i.e. have positive x component. Considering the polarization in which $\mathbf{E} \parallel \mathbf{j}, \mathbf{H} \parallel \mathbf{k}$ (the result is the same for the other polarization) we have

$$\overline{S_x} = \frac{1}{2}\,\text{Re}\; E_y(x) H_z^*(x) \tag{3}$$

$$= \frac{1}{2}\,\text{Re}\; E_y(x)\left[\frac{1}{i\omega\mu_0}\frac{dE_y^*}{dx}\right] \ge 0. \tag{4}$$

If we attempt to translate this into a condition on A_1 and A_2 using (2), we obtain a condition that **(1)** is nonlinear, **(2)** is not an equation but an inequality, and **(3)** is in fact an infinite set of conditions (since it must hold for all x) to be satisfied with only two degrees of freedom. This is clearly impossible; we are lost in a dark wood.

Floquet Solutions

The way out is to notice that, if $\epsilon_R(x)$ is periodic with period T, then it is always possible to reorganize E_1 and E_2 into two new linear combinations E^+

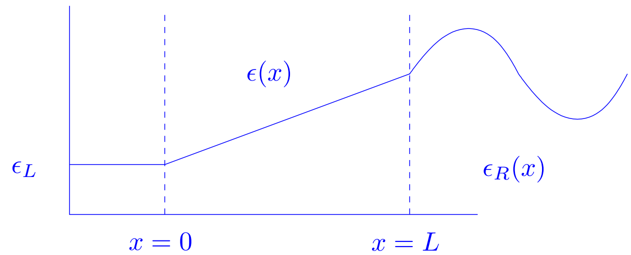

Figure 2: Linearly increasing dielectric taper function considered in Section 1.

and E^- that take the form

$$E^+(x) = e^{irx}u(x) \tag{5}$$
$$E^-(x) = e^{-irx}u(x) \tag{6}$$

where r is real, $r \geq 0$, and $u(x)$ is periodic with the same period as $\epsilon_R(x)$. The time-average Poynting vectors for these solutions are, using (3),

$$\begin{aligned}\overline{S}_+ &= \frac{r}{\omega\mu_0}|u(x)|^2 \qquad \geq 0\\ \overline{S}_- &= -\frac{r}{\omega\mu_0}|u(x)|^2 \qquad \leq 0\end{aligned}$$

and we clearly have an immediate decomposition of right- and left- moving solutions to (1). Any linear combination of the form

$$E(x) = A^+E^+(x) + A^-E^-(x)$$

is thus a solution to (1); *which* combination is again dictated by the physics of the situation, and for our situation as depicted in Figure (1), *the physics dictates simply that A^- vanishes.* This leaves us only the single degree of freedom A^+ to determine in full the electric field; this is as it must be, because the only characteristic of the electric field in the rightmost region of Figure 1 that we may choose freely is its overall magnitude, all other characteristics being determined by (1) and the requirement that the field be strictly right-moving.

Reflection Coefficient For a Linear Taper

Let us now suppose that the taper function $\epsilon(x)$ in the taper region of Figure 1 is the simple linear taper depicted in Figure 2, i.e.

$$\epsilon(x) = \epsilon_L + \frac{\epsilon_{R0} - \epsilon_L}{L}x, \tag{7}$$

where $\epsilon_{R0} = \epsilon_R(x = L)$. The function $k^2(x)$ in (1) is then

$$k^2(x) = \frac{\omega^2}{c^2}\epsilon(x) = \begin{cases} k_L^2, & x < 0 \\ k_L^2 + \frac{k_{R0}^2 - k_L^2}{L}x, & 0 \le x \le L \\ k_R(x), & L \le x \end{cases}$$

where

$$k_L^2 = \frac{\omega^2}{c^2}\epsilon_L, \qquad k_{R0}^2 = \frac{\omega^2}{c^2}\epsilon_{R0}, \qquad k_R^2(x) = \frac{\omega^2}{c^2}\epsilon_R(x).$$

As we discussed previously[2], the exact solution of (1) in the linear taper region is a linear combination of the *Hairy* functions $\text{Ha}(x)$ and $\text{Hb}(x)$, where

$$\text{Ha}(x) = \text{Ai}[-a^{1/3}(x - x_0)], \qquad \text{Hb}(x) = \text{Bi}[-a^{1/3}(x - x_0)],$$

$$a = \frac{k_{R0}^2 - k_L^2}{L}, \qquad x_0 = -\frac{k_L^2}{a}.$$

The electric field in the three regions of Figure 1 is then

$$E(x) = \begin{cases} e^{ik_L x} + re^{-ik_L x}, & x < 0 \\ \alpha\text{Ha}(x) + \beta\text{Hb}(x), & 0 \le x \le L \\ tE^+(x), & x > L \end{cases} \tag{8}$$

where $E^+(x)$ is the right-moving solution to (1) in the right region. Proceeding in lockstep with our previous treatment, we match fields and derivatives at the boundaries to obtain

$$\left(\begin{array}{c|c|c|c} -1 & \text{Ha}(0) & \text{Hb}(0) & 0 \\ \hline ik_L & \text{Ha}'(0) & \text{Hb}'(0) & 0 \\ \hline 0 & \text{Ha}(L) & \text{Hb}(L) & -E^+(L) \\ \hline 0 & \text{Ha}'(L) & \text{Hb}'(L) & -\frac{dE^+}{dx}\Big|_{x=L} \end{array}\right) \begin{pmatrix} r \\ \alpha \\ \beta \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ ik_L \\ 0 \\ 0 \end{pmatrix} \tag{9}$$

Equation (9) is the "master equation" that we solve to obtain the reflection coefficient r for the linear taper.

Summary

To summarize, the procedure for calculating the reflection coefficient for a linear taper between uniform and periodic dielectric media is as follows:

1. For a given frequency ω and periodic dielectric function $\epsilon_R(x)$, find the right- and left-moving solutions of the differential equation (1).

2. Compute the value and derivative of the right-moving solution at the left edge of the solution domain.

3. Insert these two numbers into (9) and solve for r and t.

Of course, for frequencies ω in the bandgap of the periodic medium, there are no propagating solutions, there can be no transmitted power, and we have $r = 1$ exactly.

In the following sections we will illustrate this procedure for two different choices of periodic function $\epsilon_R(x)$.

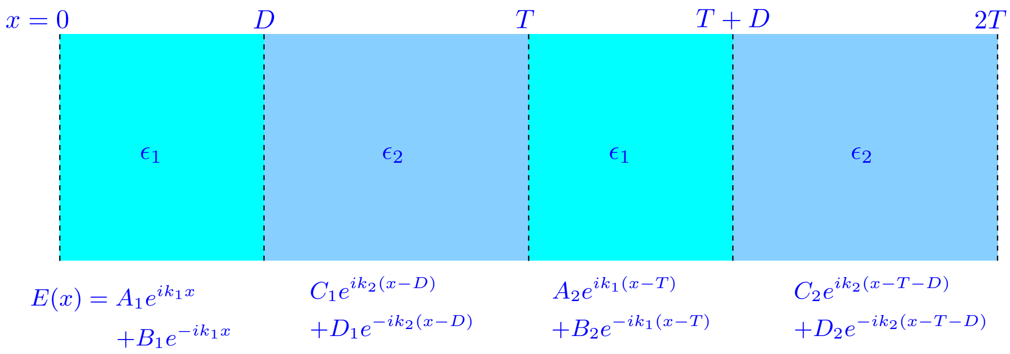

Figure 3: Step-function periodic taper.

2.2 Right-Moving Solutions for Step-Function Taper

The step-function taper is shown in Figure 3. Matching values and derivatives of $E(x)$ at $x = D$ and $x = T$, we get transfer matrices relating the coefficients in the various regions:

$$\left(\begin{array}{c} C_1 \\ D_1 \end{array}\right) = \frac{1}{2}\left(\begin{array}{cc} \left(1+\frac{k_1}{k_2}\right)e^{ik_1D} & \left(1-\frac{k_1}{k_2}\right)e^{-ik_1D} \\ \left(1-\frac{k_1}{k_2}\right)e^{ik_1D} & \left(1+\frac{k_1}{k_2}\right)e^{-ik_1D} \end{array}\right)\cdot\left(\begin{array}{c} A_1 \\ B_1 \end{array}\right) \qquad \equiv \mathbf{T_1}(\omega)\cdot\left(\begin{array}{c} A_1 \\ B_1 \end{array}\right)$$

$$\left(\begin{array}{c} A_2 \\ B_2 \end{array}\right) = \frac{1}{2}\left(\begin{array}{cc} \left(1+\frac{k_2}{k_1}\right)e^{ik_2D'} & \left(1-\frac{k_2}{k_1}\right)e^{-ik_2D'} \\ \left(1-\frac{k_2}{k_1}\right)e^{ik_2D'} & \left(1+\frac{k_2}{k_1}\right)e^{-ik_2D'} \end{array}\right)\cdot\left(\begin{array}{c} A_2 \\ B_2 \end{array}\right) \qquad \equiv \mathbf{T_2}(\omega)\cdot\left(\begin{array}{c} C_1 \\ D_1 \end{array}\right)$$

where $D' = T - D$. (Note that the ω dependence of the transfer matrices comes from their dependence on $k_i = \omega\sqrt{\epsilon_i}/c$). Putting these together, we have in general

$$\left(\begin{array}{c} A_n \\ B_n \end{array}\right) = \mathbf{T}^n(\omega)\cdot\left(\begin{array}{c} A_1 \\ B_1 \end{array}\right), \qquad \mathbf{T}(\omega) = \mathbf{T}_2(\omega)\cdot\mathbf{T}_1(\omega).$$

For ω not in a bandgap, the two eigenvalues of $\mathbf{T}(\omega)$ are a complex-conjugate pair with unit magnitude, and the eigenvector associated with the eigenvalue in the upper half-plane defines the right-traveling solution. Calling this eigenvector $\boldsymbol{\xi}^+$, and putting $\boldsymbol{\zeta}^+ = \mathbf{T}_1 \cdot \boldsymbol{\xi}^+$, the right-moving solution in the sense of the previous section is

$$E(x) = \begin{cases} \xi_1^+ e^{ik_1x} + \xi_2^+ e^{-ik_1x} & 0 \le x \le D \\ \zeta_1^+ e^{ik_1x} + \zeta_2^+ e^{-ik_1x} & D \le x \le T \end{cases}$$

and the quantities needed to evaluate (9) are

$$E^+(L) = \xi_1^+ + \xi_2^+$$

$$\left.\frac{dE^+}{dx}\right|_{x=L} = ik_1(\xi_1^+ - \xi_2^+).$$

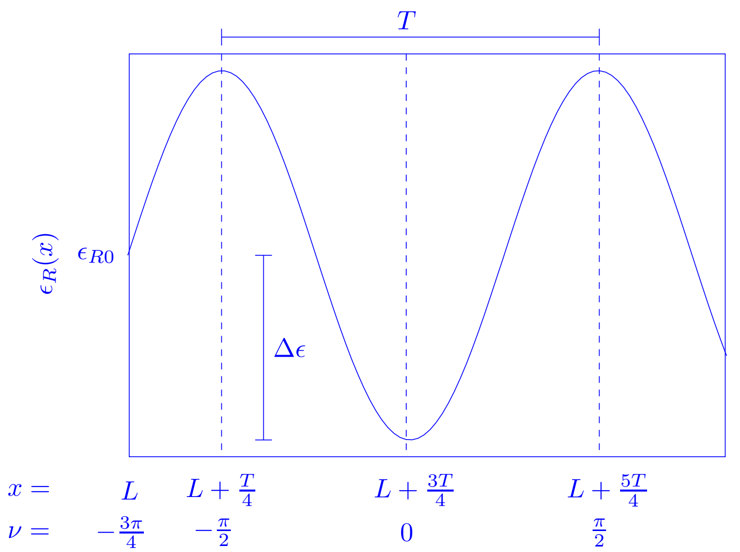

Figure 4: Sinusoidal taper.

2.3 Right-Moving Solutions for Sinusoidal Taper

The sinusoidal taper, as depicted in Figure 4, is

$$\epsilon_R(x) = \epsilon_{R0} + \Delta\epsilon \sin \frac{2\pi(x-L)}{T}.$$

Equation (1) becomes

$$\left[\frac{d^2}{dx^2} + \frac{\omega^2}{c^2}\epsilon_{R0} + \frac{\omega^2}{c^2}\Delta\epsilon \sin\frac{2\pi(x-L)}{T}\right] E(x) = 0. \tag{10}$$

With the substitutions

$$\nu = \frac{\pi(x-L)}{T} - \frac{3\pi}{4}, \qquad a = \left(\frac{\omega T}{\pi c}\right)^2 \epsilon_{R0}, \qquad q = \frac{1}{2}\left(\frac{\omega T}{\pi c}\right)^2 \Delta\epsilon$$

equation (10) becomes

$$\left[\frac{d}{d\nu^2} + a - 2q\cos(2\nu)\right] y(\nu) = 0$$

which is Mathieu's equation. The right-moving solution is of the form

$$y^+(\nu) = \text{Ce}(a,q,\nu) + i\text{Se}(a,q,\nu),$$

where Ce and Se are the standard Mathieu functions, available, for example, in MATHEMATICA. The right-moving electric field solution is then

$$E^+(x) = \text{Ce}\left(a, q, \frac{\pi}{T}(x-L) - \frac{3\pi}{4}\right) + i\text{Se}\left(a, q, \frac{\pi}{T}(x-L) - \frac{3\pi}{4}\right)$$

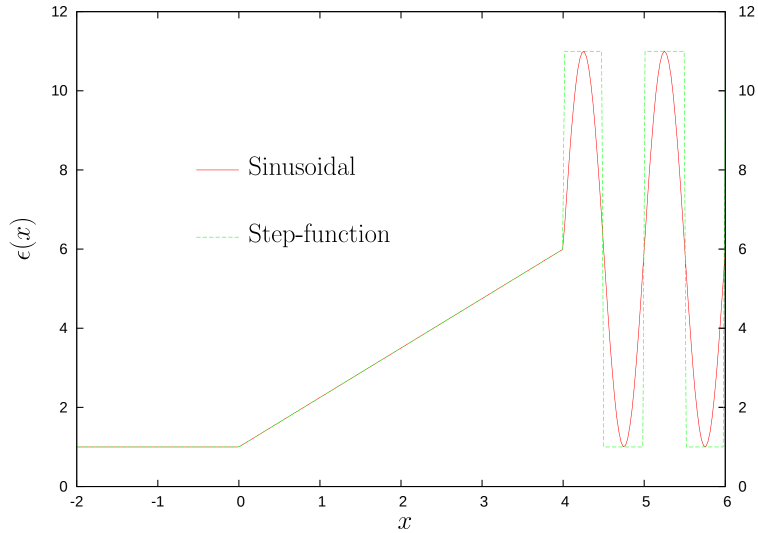

Figure 5: Linear taper to step-function and sinusoidal dielectric functions.

and the quantities needed in (9) are

$$E^+(x) = \text{Ce}\left(a, q, -\frac{3\pi}{4}\right) + i\text{Se}\left(a, q, -\frac{3\pi}{4}\right)$$
$$\left.\frac{dE^+}{dx}\right|_{x=L} = \left(\frac{\pi}{T}\right)\left[\text{Ce}'\left(a, q, -\frac{3\pi}{4}\right) + i\text{Se}'\left(a, q, -\frac{3\pi}{4}\right)\right].$$

2.4 Results

As a specific example, we consider a dielectric medium with a spatial period of $T = 1$ and maximum and minimum dielectric strengths of $\epsilon = 11$ and $\epsilon = 1$. The full dielectric function (uniform+taper+periodic) is plotted in Figure 5 for the sinusoidal and step-function cases, and the reflection coefficients for the lowest-lying frequency bands are plotted in Figures 6 and 7.

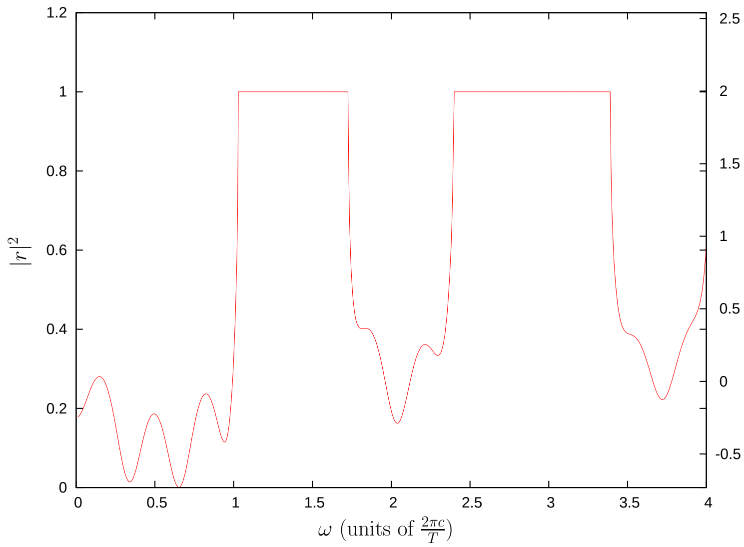

Figure 6: Reflection coefficient versus frequency for step-function taper.

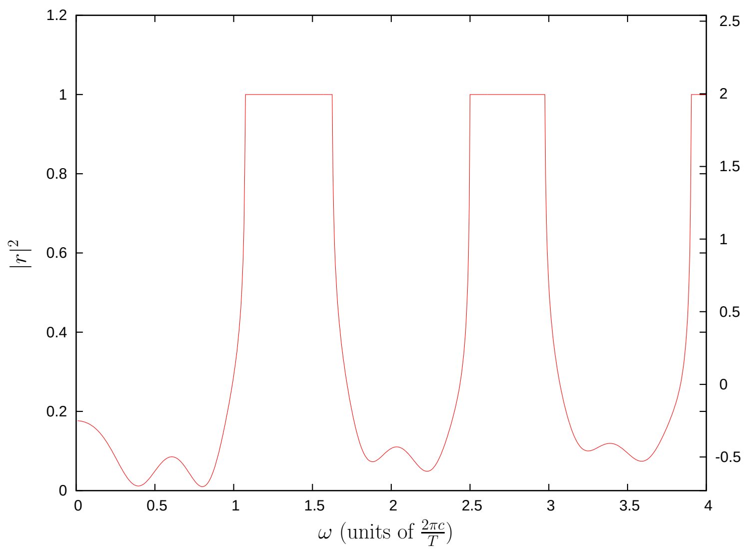

Figure 7: Reflection coefficient versus frequency for sinusoidal taper.

3 Integral Equation Method for Arbitrary Taper

3.1 Review of Green's Function Technique

Having discussed the exact solution for the *linear* taper, we next note that the fields and reflection coefficient for an *arbitrary* taper function may be computed to any desired degree of accuracy using the Green's function method of [2]. Since we discussed the method in detail in that reference, here we will discuss only the modifications necessary for application to periodic taper functions.

The Green's function method proceeds by expressing the dielectric function in the taper region as the sum of the simple linear taper (7) plus a correction term:

$$\epsilon(x) = \epsilon_{\text{lin}}(x) - \epsilon'(x), \qquad 0 \le x \le L$$

The differential equation for $E(x)$ in the taper region is then

$$\left[\frac{d^2}{dx^2} + k^2(x)\right] E(x) = 0$$

or

$$\left[\frac{d^2}{dx^2} + k_l^2(x)\right] E(x) = \Delta(x) E(x) \tag{11}$$

where, in the notation of [2],

$$k^2(x) = k_l^2(x) - \Delta(x), \qquad k_l^2 = \frac{\omega^2}{c^2}\epsilon_{\text{lin}}(x), \qquad \Delta(x) = \frac{\omega^2}{c^2}\epsilon'(x).$$

Ignoring for the moment that the RHS of (11) depends on the field $E(x)$ for which we are solving, we think of the entire RHS as a forcing term for the differential equation on the LHS, which we already know how to solve exactly. The solution to (11) is then

$$E(x) = \alpha \text{Ha}(x) + \beta \text{Hb}(x) + \int_0^L G(x,x')\Delta(x')E(x')dx' \tag{12}$$

where α and β are the *same* constants obtained from solving the master equation (9) for the linear case, and $G(x,x')$ is the solution to

$$\left[\frac{d^2}{dx^2} + k_l^2(x)\right] G(x,x') = \delta(x - x')$$

subject to the boundary conditions

$$\left| G(x,x') + \frac{1}{ik_L}\frac{d}{dx}G(x,x')\right|_{x=0} = 0 \tag{13}$$

$$\left| G(x,x') + \frac{1}{\kappa_R}\frac{d}{dx}G(x,x')\right|_{x=L} = 0. \tag{14}$$

In the second of these equations we have put

$$\kappa_R = \frac{1}{E^+(L)} \left| \frac{dE^+}{dx} \right|_{x=L}. \tag{15}$$

The quantity κ_R is the logarithmic derivative of the *right-moving* solution $E^+(x)$ in the periodic medium, evaluated at the left edge of that medium. In the case of a uniform dielectric constant $\epsilon_R(x) \equiv \epsilon_2$, we have simply $\kappa_R = ik_2$, as in equation (17) of [2]. Equation (15) is the generalization to nonconstant periodic dielectric media.

Construction of Green's Function

The construction of the Green's function proceeds exactly as before[2]. We put

$$\begin{aligned} f_1(x) &= \text{Ha}(x) + \gamma_1 \text{Hb}(x), \qquad \gamma_1 = -\left[\frac{ik_1 \text{Ha}(0) + \text{Ha}'(0)}{ik_1 \text{Hb}(0) + \text{Hb}'(0)}\right], \\ f_2(x) &= \text{Ha}(x) + \gamma_2 \text{Hb}(x), \qquad \gamma_2 = -\left[\frac{\kappa_R \text{Ha}(0) + \text{Ha}'(0)}{\kappa_R \text{Hb}(0) + \text{Hb}'(0)}\right], \end{aligned}$$

and

$$G(x, x') = \frac{1}{W} f_1(x_<) f_2(x_>) \tag{16}$$

where the Wronskian

$$W = f_1(x) f_2'(x) - f_1'(x) f_2(x)$$

may be evaluated at any point in the interval $0 \le x \le L$.

Nystrom Method

Having constructed the Green's function (16), we actually solve equation (12) using the Nystrom method. If $\{x_i, w_i\}$ is an N-point quadrature rule on the interval $[0, L]$, then the vector of values of $E(x)$ evaluated at the points $\{x_i\}$ is given by

$$\mathbf{E} = \Big(\mathbb{1} - \mathbf{K}\Big)^{-1} \cdot \Big[\alpha \mathbf{Ha} + \beta \mathbf{Hb}\Big],$$

$$\mathbf{Ha}_i = \text{Ha}(x_i), \qquad \mathbf{Hb}_i = \text{Hb}(x_i), \qquad K_{ij} = G(x_i, x_j)\Delta(x_j) w_j.$$

After solving for $\mathbf{E}_i$, we can find the electric field anywhere by evaluating (12) using the quadrature rule:

$$E(x) = \alpha \text{Ha}(x) + \beta \text{Hb}(x) + \sum_i G(x, x_i)\Delta(x_i) E(x_i) w_i.$$

In particular, $E(0)$ may be obtained this way, and the reflection coefficient calculated simply according to

$$r = E(0) - 1.$$

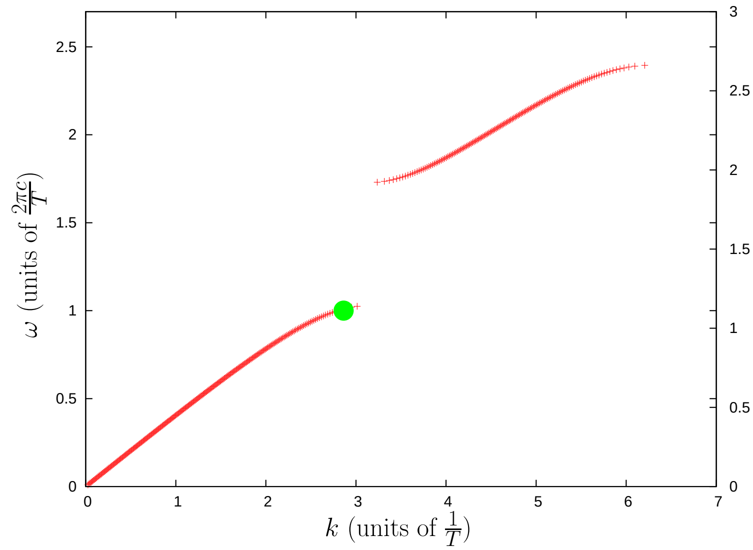

Figure 8: First two frequency bands for step-function medium. We choose the point labeled by the green dot ($\omega = 1.01 \cdot \frac{2\pi c}{T}$) as the frequency at which to optimize our taper design.

This has all been a whirlwind review of the Green's function technique, but we urge the interested reader to verify that (12), with the Green's function defined by (16), **(1)** satisfies equation (11) in the taper region and **(2)** upholds the essential requirement that there be no right-moving wave in the periodic medium; this latter condition is ensured by (14).

3.2 Application: Single-Frequency Taper Optimization

As an application of the Green's function technique, let's see how far we can get with a simple single-term correction to the linear taper. We put

$$\Delta(x) = \Delta_1 \sin \frac{\pi x}{L},$$

so that the full taper function is

$$k^2(x) = k_l^2(x) - \Delta_1 \sin \frac{\pi x}{L}, \tag{17}$$

and we tweak Δ_1 to see how small we can make the reflection coefficient with this simple correction. We consider the case of the step-function periodic medium, for which the lowest two bands are shown in Figure 8; as shown in the figure, we work at frequency $\omega = 1.01 \cdot \frac{2\pi c}{T}$, which is just below the band edge and yields a poor result ($|r|^2 \approx 0.429$) with the simple linear taper. Figure 9 shows a plot of $|r|^2$ versus the parameter Δ_1 in (17). There is a clear minimum at $\Delta_1 \approx 0.7$, but

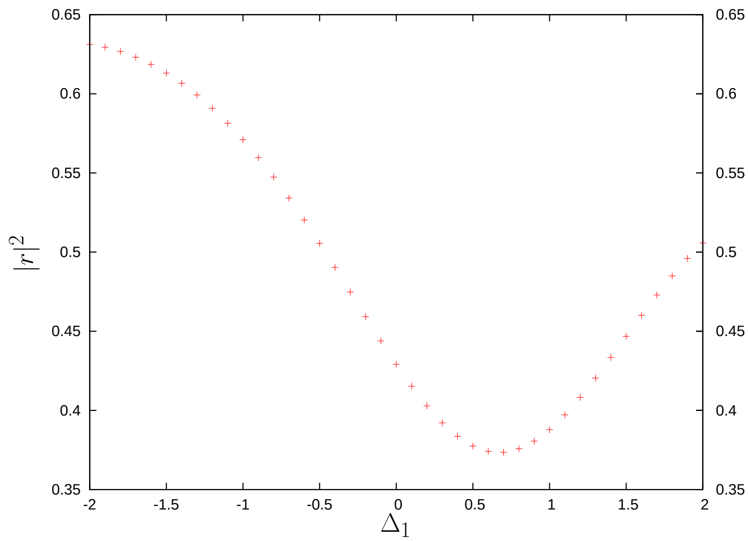

Figure 9: Reflection coefficient calculated for various values of the Δ_1 parameter in (17).

this only gets us down to $|r|^2 \approx 0.37$. To improve further, we add additional terms of the form $\Delta_n \sin\left(\frac{n\pi x}{L}\right)$ to (17), running a separate 1D optimization problem for each of the parameters Δ_n. By adding three additional terms to (17) we are able to reduce the reflection coefficient to less than $1.0{\cdot}10^{-3}$, as shown in Figure 10. Our optimal taper is

$$\epsilon(x) = \epsilon_l(x) - \sum_{n=1}^{4} \epsilon_n \sin\left(\frac{n\pi x}{L}\right), \qquad (18)$$

where $\epsilon_l(x)$ is the linear taper (7), and where

$$\begin{aligned} \epsilon_1 &\approx 0.69 \\ \epsilon_2 &\approx -5.6 \\ \epsilon_3 &\approx -0.20 \\ \epsilon_4 &\approx +1.4. \end{aligned}$$

This optimal taper design is plotted in (11).

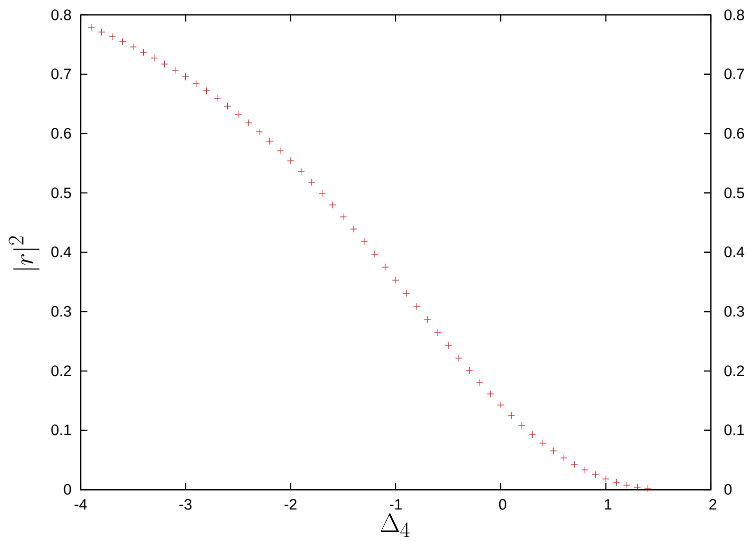

Figure 10: Final stage of iterative one-dimensional taper optimization procedure. Plotted is the reflection coefficient versus Δ_4, where the full taper function is $k^2(x) = k_l^2 - \sum_{n=1}^{4} \Delta_n \sin\left(\frac{n\pi x}{L}\right)$ and the optimal values of the other coefficients have been previously determined to be $\Delta_1 = 0.7$, $\Delta_2 = -5.7$, and $\Delta_3 = -0.2$. From this plot, the optimal value of Δ_4 is around 1.4, at which point the reflection coefficient is less than $1.0 \cdot 10^{-3}$.

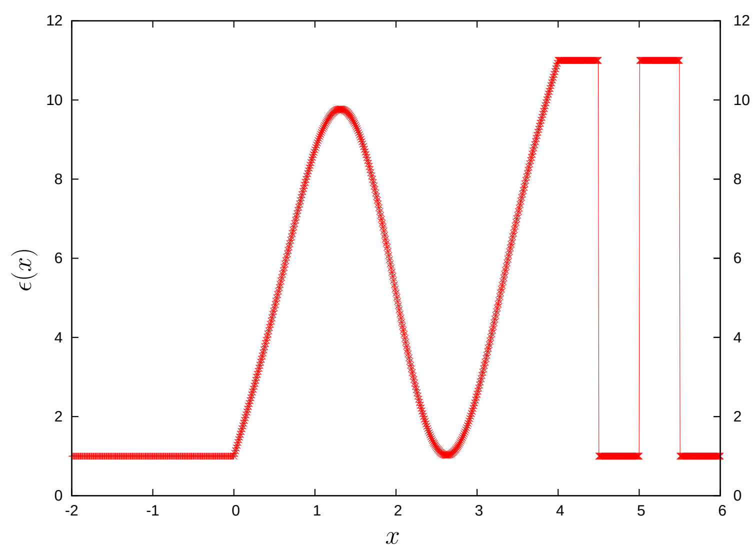

Figure 11: Optimal taper function (18) as determined by simple optimization procedure discussed in text.

References

[1] H. Reid, first memo in this series.

[2] H. Reid, second memo in this series.