How Does the Boundary-Element Method Work?

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The boundary-element method (BEM) often proves more efficient and more accurate than finite-element or finite-difference techniques for solving electrostatic problems. But the large number of terms appearing in the discretized BEM equations make the method seem mysterious to beginners and seasoned researchers alike. To develop intuition for the various terms in the equations, and to see how the BEM works in detail, it is thus useful to apply the method to a simple, analytically solvable test problem.

Problem Geometry

We consider a point charge at the center of a dielectric sphere, as depicted in Figure 1. We take the origin of coordinates at the center of sphere, call the radius of the sphere R and its dielectric constant ϵ_1, and seek to determine the electrostatic potential $\phi(\mathbf{r})$ at all points in space.

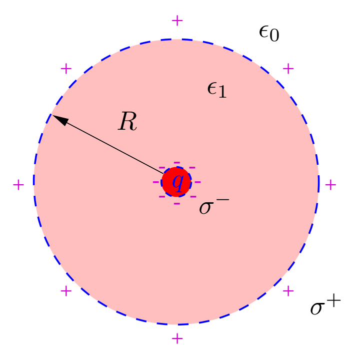

Figure 1: Point charge q embedded in dielectric sphere. Bound polarization charge densities σ^- and σ^+ develop on the inner and outer surfaces of the sphere. (The "inner surface" of the sphere is defined by imagining the point charge to have some finite radius, which we eventually take to 0.)

Analytical Solution

The solution to this problem is

$$\phi(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_1 r} + \Phi_0, & r \le R \\ \\ \dfrac{q}{4\pi\epsilon_0 r}, & r \ge R. \end{cases} \tag{1}$$

The first term in the expression for $r \le R$ is the potential of the point charge screened by the bound polarization charge σ^- that develops on the inner surface of the sphere. The constant ϕ_0 is the contribution of the bound polarization charge σ^+ that develops on the *outer* surface of the sphere (the potential of a uniform spherical shell of charge being constant inside the sphere). Although we don't know *a priori* what σ^+ is, we can find ϕ_0 from the requirement that ϕ be continuous at $r = R$:

$$\begin{aligned} \phi_0 &= \frac{q}{4\pi\epsilon_0 R} - \frac{q}{4\pi\epsilon_1 R} \\ &\equiv \left(\frac{\epsilon_1 - \epsilon_0}{\epsilon_1}\right)\phi_R \end{aligned}$$

where we defined the shorthand notation

$$\phi_R \equiv \frac{q}{4\pi\epsilon_0 R} \tag{2}$$

for the value of the potential at the surface of the sphere.

For $r \ge R$, ϕ is just the unscreened field of the point charge, since the contributions of σ^- and σ^+ cancel outside the sphere.

The electric field is

$$E_r(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\epsilon_1 r^2}, & r \le R \\ \\ \dfrac{q}{4\pi\epsilon_0 r^2}, & r \ge R. \end{cases}$$

Introducing the notation $E_R^{\text{in,out}}$ for the inner and outer fields at the boundary of the sphere, we have

$$E_R^{\text{in}} = \frac{q}{4\pi\epsilon_1 R^2}, \qquad E_R^{\text{out}} = \frac{q}{4\pi\epsilon_0 R^2} \tag{3}$$

and the discontinuity condition

$$\epsilon_1 E_R^{\text{in}} = \epsilon_0 E_R^{\text{out}}$$

is satisfied.

BEM Solution

In the general BEM formulation, we use Green's theorem to write the electrostatic potential in region i as

$$\phi_i(\mathbf{r}) = \frac{1}{4\pi}\oint\left\{G(\mathbf{r},\mathbf{r}')\frac{\partial\phi}{\partial n}(\mathbf{r}') - \frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}')\right\}dA + \phi_i^{\text{bare}}(\mathbf{r})$$

where the surface integral is over the bounding surface of region i, $\frac{\partial}{\partial n}$ is the component of the gradient in the direction of the outward-pointing normal to that surface, ϕ_i^{bare} is the potential due to all charges contained in the bulk of region i, and

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

is the Green's function for the Poisson equation.

In the case at hand, there are just two regions ($i = 1, 2$). In the inner region (region 1), the outward-pointing surface normal is $+\hat{r}$, and ϕ^{bare} is just the field of the screened point charge. In the outer region (region 2), the outward-pointing surface normal is $-\hat{r}$, and ϕ^{bare} vanishes, as there are no charges in the outer region. We can then write the fields in the two regions as

$$\begin{aligned}
\phi_1(\mathbf{r}) &= -\frac{1}{4\pi}\oint\left\{G(\mathbf{r},\mathbf{r}')E_R^{\text{in}} + \frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}')\phi_R\right\}dA + \frac{q}{4\pi\epsilon_1 r}\\
\phi_2(\mathbf{r}) &= +\frac{1}{4\pi}\oint\left\{G(\mathbf{r},\mathbf{r}')E_R^{\text{out}} + \frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}')\phi_R\right\}dA.
\end{aligned}$$

Because E_R and ϕ_R are constant over the surface of the sphere, we may pull them out of the integrals to find

$$\begin{aligned}
\phi_1(\mathbf{r}) &= -\left\{\mathbb{G}(\mathbf{r})E_R^{\text{in}} + \mathbb{N}(\mathbf{r})\phi_R\right\} + \frac{q}{4\pi\epsilon_1 r}\\
\phi_2(\mathbf{r}) &= +\left\{\mathbb{G}(\mathbf{r})E_R^{\text{out}} + \mathbb{N}(\mathbf{r})\phi_R\right\}.
\end{aligned}$$

where we put

$$\mathbb{G}(\mathbf{r}) = \frac{1}{4\pi}\oint G(\mathbf{r},\mathbf{r}')dA, \qquad \mathbb{N}(\mathbf{r}) = \frac{1}{4\pi}\oint\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}')dA.$$

The first integral is simple to evaluate:

$$\mathbb{G}(\mathbf{r}) = \frac{1}{4\pi}\oint\frac{R^2 d\Omega}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{2}\int_0^\pi \frac{R^2\sin\theta d\theta}{\sqrt{r^2+R^2-2rR\cos\theta}} = \begin{cases} R, & r \le R\\ R^2/r, & r\ge R\end{cases} \tag{4}$$

To evaluate the second integral, we note that

$$\frac{\partial G}{\partial n}(\mathbf{r},\mathbf{r}') = \nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\cdot\hat{\mathbf{r}}' = \frac{(\mathbf{r}-\mathbf{r}')\cdot\hat{\mathbf{r}}'}{|\mathbf{r}-\mathbf{r}'|^{3/2}}$$

so

$$\mathbb{N}(\mathbf{r}) = \frac{1}{2}\int_0^\pi \frac{(rR\cos\theta - R^2)R\sin\theta d\theta}{(r^2+R^2-2rR\cos\theta)^{3/2}} = \begin{cases} -1, & r \le R \\ 0, & r \ge R. \end{cases} \tag{5}$$

Combining (4) and (5) with the previously obtained results (2) and (3), we have

$$\begin{aligned} \phi_1(\mathbf{r}) &= -\left\{ \underbrace{\mathbb{G}(\mathbf{r})E_R^{\text{in}}}_{=\frac{q}{4\pi\epsilon_1 R}} + \underbrace{\mathbb{N}(\mathbf{r})\phi_R}_{=-\frac{q}{4\pi\epsilon_0 R}} \right\} + \frac{q}{4\pi\epsilon_1 r} \\ &= \left(\frac{\epsilon_1-\epsilon_0}{\epsilon_1}\right)\frac{q}{4\pi\epsilon_1 R} + \frac{q}{4\pi\epsilon_1 r} \end{aligned}$$

and

$$\begin{aligned} \phi_2(\mathbf{r}) &= +\left\{ \underbrace{\mathbb{G}(\mathbf{r})E_R^{\text{out}}}_{=\frac{q}{4\pi\epsilon_0 r}} + \underbrace{\mathbb{N}(\mathbf{r})\phi_R}_{=0} \right\} \\ &= \frac{q}{4\pi\epsilon_0 r} \end{aligned}$$

in agreement with (1).

The relevant points here are

- In the interior region, $\mathbb{G}$-type terms in the surface integral yield a *negative* contribution to the potential, while $\mathbb{N}$-type terms give a *positive* contribution. This statement is generally true for interior regions containing net positive charges (all signs are flipped if the net charge is negative). In this particular high-symmetry case, both terms in the surface integral happen to evaluate to constants, but this will not be true for more general situations.

- In the exterior region, $\mathbb{G}$-type terms in the surface integral give the dominant contribution to the potential, and this contribution is *positive* for net positive charge in the interior region. (In this particular high-symmetry case, the $\mathbb{N}$-type terms happen to give *no* contribution to the exterior potential. This will not be true in more general situations, but it is generally true that $\mathbb{G}$-type terms give larger contributions than $\mathbb{N}$-type terms in exterior regions.).

Figure 2 shows the various contributions to the BEM expressions for the potential in the two regions.

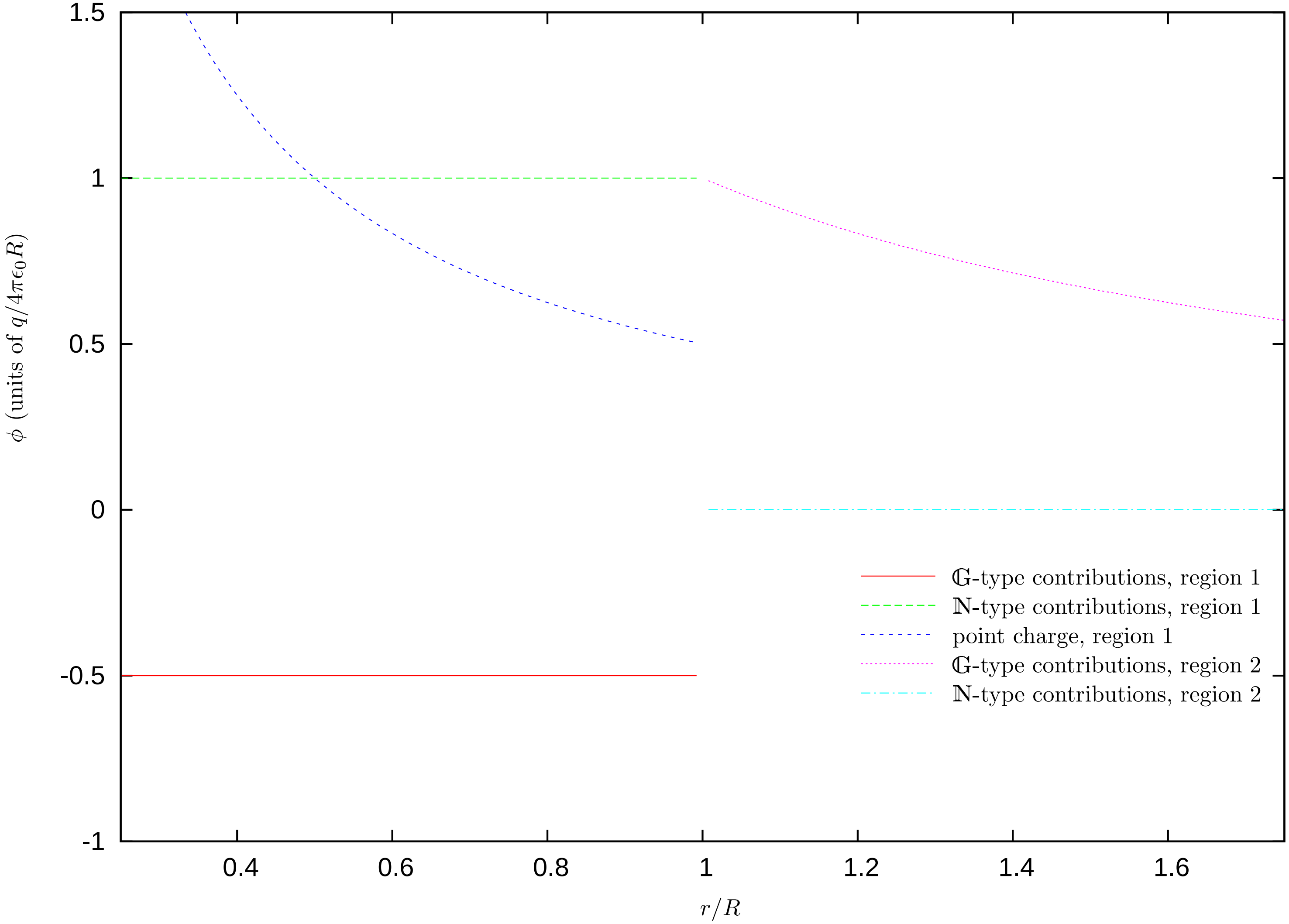

Figure 2: The various contributions to the potential in the different regions for the case $\epsilon_1 = 2\epsilon_0$.