Chapter 5: Problems 10-18

Problem 5.10

A circular current loop of radius $a$ carrying a current $I$ lies in the $x - y$ plane with its center at the origin.

(a) Show that the only nonvanishing component of the vector potential is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{\pi} \int_0^\infty dk \cos kz I_1(k \rho_<)K_1(k \rho_<)$$

where $\rho_<(\rho_>)$ is the smaller (larger) of $a$ and $\rho$.

(b) Show that an alternative expression for $A_\phi$ is

$$A_\phi(\rho, z) = \frac{\mu_0 I a}{2} \int_0^\infty dk e^{-k|z|} J_1(ka)J_1(k\rho).$$

(c) Write down integral expressions for the components of magnetic induction, using the expressions of parts a and b. Evaluate explicitly the components of $\mathbf{B}$ on the $z$ axis by performing the necessary integrations.

(a) Translating Jackson’s equation (5.33) into cylindrical coordinates, we have

$$J_\phi = I \delta(z) \delta(\rho - a) \tag{1}$$

Following Jackson, we take the observation point $x$ on the $x$ axis, so its coordinates are $(\rho, \phi = 0, z)$. Since there is no current in the $z$ direction, and since the
current density is cylindrically symmetric, there is no vector potential in the $\rho$ or $z$ directions. In the $\phi$ direction we have

$$A_\phi = -A_z \sin \phi + A_y \cos \phi = A_y$$

$$= \frac{\mu_0}{4\pi} \int \frac{J_y(x')}{|x - x'|} \, dx'$$

$$= \frac{\mu_0}{4\pi} \int \frac{J_\phi(x') \cos \phi'}{|x - x'|} \, dx'$$

$$= \frac{\mu_0}{4\pi} \text{Re} \int \frac{J_\phi(x') e^{i\phi'}}{|x - x'|} \, dx'$$

$$= \frac{\mu_0}{4\pi} \text{Re} \int J_\phi(x') e^{i\phi'} \left[ \sum_{m=-\infty}^{\infty} \int_0^\infty e^{i(m\phi - \phi')} \cos[k(z - z')] I_m(k\rho_<) K_m(k\rho_> \, dk \right] \, dx'$$

where we substituted in Jackson’s equation (3.148). Rearranging the order of integration and remembering that $\phi = 0$, we have

$$A_\phi = \frac{\mu_0}{2\pi^2} \text{Re} \sum_{m=-\infty}^{\infty} \int_0^\infty \left[ \int J_\phi(x') e^{i(1-m)\phi'} \cos[k(z - z')] I_m(k\rho_<) K_m(k\rho_> \, dx' \right] \, dk$$

If $m = 1$, the $\phi$ integral yields $2\pi$; otherwise it vanishes. Thus

$$A_\phi = \frac{\mu_0}{\pi} \int_0^{\infty} \left[ \int_0^{\infty} \int_{-\infty}^{\infty} J_\phi(r', z') \cos[k(z - z')] I_1(k\rho_<) K_1(k\rho_> \rho' \, dz' \, dr' \right] \, dk$$

Substituting (1), we have

$$A_\phi = \frac{I a \mu_0}{\pi} \int_0^{\infty} \cos k z I_1(k\rho_<) K_1(k\rho_> \, dk$$

(b) The procedure for obtaining this expression is identical to the one I just went through, but with the expression from Problem 3.16(b) used for the Green’s function instead of equation (3.148).

(c) Let’s suppose that the observation point is in the interior region of the current loop, so $\rho_< = \rho, \rho_> = a$. Then

$$B_\rho = [\nabla \times A]_\rho = -\frac{\partial A_\phi}{\partial z}$$

$$= -\frac{I a \mu_0}{\pi} \int_0^{\infty} k \sin k z I_1(k\rho) K_1(ka) \, dk$$

$$B_z = [\nabla \times A]_z = \frac{1}{\rho} A_\phi + \frac{\partial A_\phi}{\partial \rho}$$

$$= \frac{I a \mu_0}{\pi} \int_0^{\infty} \cos k z \left[ \frac{I_1(k\rho)}{\rho} + k I_1'(k\rho) \right] K_1(ka) \, dk$$
As \( r = 0 \), \( I_1(\rho) \to 0 \), \( I_1(\rho)/\rho \to 1/2 \), and \( I'_1(\rho) \to 1/2 \), so

\[
B_r(\rho = 0) = 0
\]

\[
B_z(\rho = 0) = \frac{Ia\mu_0}{\pi} \int_0^\infty k \cos k_1(ka) \, dk
= \frac{Ia\mu_0}{\pi} \frac{\partial}{\partial z} \int_0^\infty \sin k_1(ka) \, dk
\]

The integral may be done by parts:

\[
\int_0^\infty \sin k_1(kz) \, dk = \left[ -\frac{1}{a} \sin k_1(ka) \right]_0^\infty + \frac{z}{a} \int_0^\infty \cos k_1(ka) \, dk
\]

\( K_0 \) is finite at zero but \( \sin \) vanishes there, and \( \sin \) is finite at infinity but \( K_0 \) vanishes there, so the first term vanishes. The integral in the second term is Jackson’s equation (3.150). Plugging it in to the above,

\[
B_z(\rho = 0) = \frac{Ia\mu_0}{2} \frac{\partial}{\partial z} \frac{z}{(z^2 + a^2)^{1/2}}
= \frac{Ia\mu_0}{2} \frac{a^2}{(z^2 + a^2)^{3/2}}.
\]

**Problem 5.11**

A circular loop of wire carrying a current \( I \) is located with its center at the origin of coordinates and the normal to its plane having spherical angles \( \theta_0, \phi_0 \). There is an applied magnetic field, \( B_x = B_0(1 + \beta y) \) and \( B_y = B_0(1 + \beta x) \).

(a) Calculate the force acting on the loop without making any approximations. Compare your result with the approximate result (5.69). Comment.

(b) Calculate the torque in lowest order. Can you deduce anything about the higher order contributions? Do they vanish for the circular loop? What about for other shapes?

(a) Basically we’re dealing with two different reference frames here. In the “lab” frame, \( \mathcal{R} \), the magnetic field exists only in the \( xy \) plane, and the normal to the current loop has angles \( \theta_0, \phi_0 \). We define the “rotated” frame \( \mathcal{R}' \) by aligning the \( z' \) axis with the normal to the current loop, so that in \( \mathcal{R}' \) the current loop exists only in the \( x'y' \) plane, but the magnetic field now has a \( z' \) component.

The force on the current loop is

\[
\mathbf{F} = \int (\mathbf{J} \times \mathbf{B}) \, dV.
\]
The components of $J$ are easy to express in $R'$, but more complicated in $R$; the opposite is true for $B$. There are two ways to do the problem: we can work out the components of $J$ in $R$ and do the integral in $R$, or we can work out the components of $B$ in $R'$ and do the integral in $R'$, in which case we would have to transform the components of the force back to $R$ to get the answer we desire. I think the former approach is easier.

To derive the transformation matrix relating the coordinates of a point in $R$ and $R'$, I imagined that the transformation arose from two separate transformations, as depicted in figure 1. The first transformation is a rotation through $\phi_0$ around the $z$ axis, which takes us from $R$ to an intermediate frame $R_1$. Then we rotate through $\theta_0$ around the $y_1$ axis, which takes us to $R'$. Evidently, the coordinates of a point in the various frames are related by

$$
\begin{pmatrix}
  x_1 \\
  y_1 \\
  z_1
\end{pmatrix} =
\begin{pmatrix}
  \cos \phi_0 & \sin \phi_0 & 0 \\
  -\sin \phi_0 & \cos \phi_0 & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} \quad (3)
$$

$$
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix} =
\begin{pmatrix}
  \cos \theta_0 & 0 & -\sin \theta_0 \\
  0 & 1 & 0 \\
  \sin \theta_0 & 0 & \cos \theta_0
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  y_1 \\
  z_1
\end{pmatrix} \quad (4)
$$

Multiplying matrices,

$$
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix} =
\begin{pmatrix}
  \cos \theta_0 \cos \phi_0 & \cos \theta_0 \sin \phi_0 & -\sin \theta_0 \\
  -\sin \phi_0 & \cos \phi_0 & 0 \\
  \sin \theta_0 \cos \phi_0 & \sin \theta_0 \sin \phi_0 & \cos \theta_0
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}. \quad (5)
$$

This matrix also gives us the transformation between unit vectors in the two
frames:

\[
\begin{pmatrix}
\hat{i}' \\
\hat{j}' \\
\hat{k}'
\end{pmatrix} = \begin{pmatrix}
\cos \theta_0 \cos \phi_0 & \cos \theta_0 \sin \phi_0 & -\sin \theta_0 \\
-\sin \phi_0 & \cos \phi_0 & 0 \\
\sin \theta_0 \cos \phi_0 & \sin \theta_0 \sin \phi_0 & \cos \theta_0
\end{pmatrix}
\begin{pmatrix}
\hat{i} \\
\hat{j} \\
\hat{k}
\end{pmatrix}.
\]

(6)

We will also the inverse transformation, i.e. the expressions for coordinates in \(R\) in terms of coordinates in \(R'\):

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} = \begin{pmatrix}
\cos \theta_0 \cos \phi_0 & -\sin \phi_0 & \sin \theta_0 \cos \phi_0 \\
\cos \theta_0 \sin \phi_0 & \cos \phi_0 & \sin \theta_0 \sin \phi_0 \\
-\sin \theta_0 & 0 & \cos \theta_0
\end{pmatrix}
\begin{pmatrix}
x' \\
y' \\
z'
\end{pmatrix}.
\]

(7)

To do the integral in (2) it’s convenient to parameterize a point on the current loop by an angle \(\phi'\) reckoned from the \(x'\) axis in \(R'\). If the loop radius is \(a\), then the coordinates of a point on the loop are \(x' = a \cos \phi'\), \(y' = a \sin \phi'\), and the current density/volume element product is

\[
J \, dV = Idl = (Ia \, d\phi') \hat{\phi}'
\]

\[
= Ia \, d\phi' \left[ -\sin \phi' \hat{j} + \cos \phi' \hat{k} \right]
\]

\[
= Ia \, d\phi' \left[ (-\sin \phi' \cos \theta_0 \cos \phi_0 - \cos \phi' \sin \phi_0) \hat{i}
\]

\[
+ (\sin \phi' \sin \phi_0 + \cos \phi' \cos \phi_0) \hat{j} + (\sin \phi' \sin \theta_0) \hat{k} \right]
\]

We also need the components of the \(B\) field at a point on the current loop:

\[
B(\phi') = B_0[1 + \beta y(\phi')] \hat{i} + B_0[1 + \beta x(\phi')]
\]

\[
= B_0[1 + a \beta (\cos \phi' \cos \theta_0 \sin \phi_0 + \sin \phi' \cos \phi_0)] \hat{i} + B_0[1 + a \beta (\cos \phi' \cos \theta_0 \cos \phi_0 - \sin \phi' \sin \phi_0)] \hat{j}
\]

The components of the cross product are

\[
[J \times B]_x \, dV = -J_z B_y \, dV
\]

\[
= (\cdots) \beta Ia^2 B_0 \, d\phi' \left( \sin^2 \phi' \sin \theta_0 \sin \phi_0 \right)
\]

\[
[J \times B]_y \, dV = J_z B_x \, dV
\]

\[
= (\cdots) + \beta Ia^2 B_0 \, d\phi' \left( \sin^2 \phi' \sin \theta_0 \cos \phi_0 \right)
\]

\[
[J \times B]_z \, dV = (J_y B_x - J_x B_y) \, dV
\]

\[
= (\cdots) + 0
\]

where we only wrote out terms containing a factor of \(\cos^2 \phi'\) or \(\sin^2 \phi'\), since only these terms survive after the integral around the current loop (we grouped all the remaining terms into \((\cdots)\)). In the surviving terms, \(\cos^2 \phi'\) and \(\sin^2 \phi'\) turn into factors of \(\pi\) after the integral around the loop. Then the force components are

\[
F_x = \pi \beta Ia^2 B_0 \sin \theta_0 \sin \phi_0
\]

\[
F_y = \pi \beta Ia^2 B_0 \sin \theta_0 \cos \phi_0
\]

\[
F_z = 0.
\]
To compare this with the first-order approximate result, note that the magnetic
moment has magnitude \( \pi a^2 I \) and is oriented along the \( z' \) axis:

\[
\mathbf{m} = \pi a^2 I \mathbf{k}' = \pi a^2 I \left( \sin \theta_0 \cos \phi_0 \mathbf{i} + \sin \theta_0 \sin \phi_0 \mathbf{j} + \cos \theta_0 \mathbf{k} \right)
\]

so

\[
\nabla (\mathbf{B} \cdot \mathbf{m}) = \nabla \left( B_0 (1 + \beta y) m_x + B_0 (1 + \beta x) m_y \right) = B_0 \beta \left( m_y \mathbf{i} + m_x \mathbf{j} \right) = \pi \beta I a^2 B_0 \left( \sin \theta_0 \sin \phi_0 \mathbf{i} + \sin \theta_0 \cos \phi_0 \mathbf{j} \right)
\]

in exact agreement with the result we calculated so laboriously above.

**Problem 5.12**

Two concentric circular loops of radii \( a, b \) and currents \( I, I' \), respectively \((b < a)\), have an angle \( \alpha \) between their planes. Show that the torque on one of the loops
is about the line of intersection of the two planes containing the loops and has the
magnitude

\[
N = \frac{\mu_0 \pi I I' b^2}{2a} \sum_{n=0}^{\infty} \frac{(n+1)}{(2n+1)} \frac{\Gamma(n+3/2)}{\Gamma(n+2)\Gamma(3/2)} \left( \frac{b}{a} \right)^{2n} P_{2n+1}(\cos \alpha).
\]

The torque on the smaller loop is

\[
N = \int \mathbf{r} \times \left( \mathbf{J}_b(\mathbf{r}) \times \mathbf{B}_a(\mathbf{r}) \right) d\mathbf{r}
= \int \left\{ [\mathbf{r} \cdot \mathbf{B}_a(\mathbf{r})] \mathbf{J}_b(\mathbf{r}) - [\mathbf{r} \cdot \mathbf{J}_b(\mathbf{r})] \mathbf{B}_a(\mathbf{r}) \right\} d\mathbf{r}.
\]

where \( \mathbf{J}_b \) is the current density of the smaller loop and \( \mathbf{B}_a \) is the magnetic field of the larger loop. But \( \mathbf{r} \cdot \mathbf{J}_b \) vanishes, because the current flows in a circle around the origin—there is no current flowing toward or away from the origin. Thus

\[
N = \int r B_r(\mathbf{r}) \mathbf{J}_b(\mathbf{r}) d\mathbf{r}
\]  

(8)

where \( B_r \) is the radial component of the magnetic field of the larger current loop.

As in the last problem, it’s convenient to define two reference frames for this situation. Let \( \mathcal{R} \) be the frame in which the smaller loop (radius \( b \), current \( I \))
lies in the \( xy \) plane, and \( \mathcal{R}' \) the frame in which the larger loop lies in the \( x'y' \) plane. We might as well take the line of intersection of the two planes to be the \( y \) axis, so \( y = y' \). Then the \( z' \) axis has spherical coordinates \((\theta = \alpha, \phi = 0)\) in
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\(R_\text{0}\), and for transforming back and forth between the two frames we may use the transformation matrices we derived in the last problem, with \(\theta_0 = \alpha, \phi_0 = 0\). If we choose to evaluate the integral (8) in frame \(R\), the current density is

\[
J_b(r) = I \delta(r - b) \delta(\theta - \pi/2) \left[ - \sin \phi \hat{i} + \cos \phi \hat{j} \right]
\]

so the components of the torque are

\[
N_x = -I b^2 \int_0^{2\pi} B_r(r = b, \theta = \pi/2, \phi) \sin \phi \, d\phi
\]

(9)

\[
N_y = I b^2 \int_0^{2\pi} B_r(r = b, \theta = \pi/2, \phi) \cos \phi \, d\phi
\]

(10)

To do the integral in (8), we need an expression for the radial component \(B_r\) of the field of the larger loop. Of course, we already have an expression for the field in \(R_\text{0}\): in that frame the field is just that of a circular current loop in the \(x'y'\) plane, Jackson’s equation (5.48):

\[
B^r_{r'}(r', \theta') = \frac{\mu_0 I a}{2r'} \sum_{l=0}^{\infty} \frac{(-1)^l(l+1)!!}{2^l l!!} r_{<}^{2l+1} P_{2l+1}(\cos \theta').
\]

(11)

We are interested in evaluating this field at points along the smaller current loop, and for all such points \(r = b\); then \(r_\prec = b, r_\succ = a\) and we have

\[
B^r_{r'}(r' = b, \theta') = \frac{\mu_0 I}{2a} \sum_{l=0}^{\infty} \frac{(-1)^l(l+1)!!}{2^l l!!} \left( \frac{b}{a} \right)^{2l} P_{2l+1}(\cos \theta').
\]

(11)

To transform this to frame \(R\), we first note that, since the origins of \(R\) and \(R'\) coincide, the unit vectors \(\hat{r}\) and \(\hat{r}'\) coincide, so \(B_r = B^r_{r'}\). Next, (11) expresses the field in terms of \(\cos \theta'\), the polar angle in frame \(R'\). How do we write this in terms of the angles \(\theta\) and \(\phi\) in frame \(R\)? Well, note that

\[
\cos \theta' = \frac{x' \sin \alpha + z \cos \alpha}{r}
\]

\[
= \frac{r \sin \theta \cos \phi \sin \alpha + r \cos \theta \cos \alpha}{r}
\]

\[
= \sin \theta \sin \alpha \cos \phi + \cos \theta \cos \alpha
\]

(12)

where in the second line we used the transformation matrix from Problem 5.11 to write down \(z'\) in terms of \(x\) and \(z\). Equation (12) is telling us what our coordinates in \(R'\) are in terms of our coordinates in \(R\); if a point has angular coordinates \(\theta, \phi\) in \(R\), then (12) tells us what angle \(\theta'\) it has in \(R'\). (We could also work out what the azimuthal angle \(\phi'\) would be, but we don’t need to, because (11) doesn’t depend on \(\phi'\).)
To express the Legendre function in (11) with the argument (12), we may make use of the addition theorem for associated Legendre polynomials:

\[ P_l(\cos \theta') = P_l(\cos \theta \cos \alpha + \sin \theta \sin \alpha \cos \phi) \]

\[ = P_l(\cos \theta)P_l(\cos \alpha) + 2 \sum_{m=1}^{l} P_m^m(\cos \theta)P_l^m(\cos \alpha) \cos m \phi. \]

Of course, the smaller loop exists in the \( xy \) plane, so for all points on that loop we have \( \theta = \pi/2 \), whence

\[ P_l(\cos \theta') = P_l(0)P_l(\cos \alpha) + 2 \sum_{m=1}^{l} P_m^m(0)P_l^m(\cos \alpha) \cos m \phi. \]

We may now write down an expression for the radial component of the magnetic field of the larger loop, evaluated at points on the smaller loop, in terms of the angle \( \phi \) that goes from 0 to \( 2\pi \) around that loop:

\[ B_r(\phi) = \frac{\mu_0 I}{2a} \sum_{l=0}^{\infty} \frac{(-1)^l(2l + 1)!!}{2^l l!} \left( \frac{b}{a} \right)^{2l} \left\{ P_{2l+1}(0)P_{2l+1}(\cos \alpha) \right. \]

\[ \left. + 2 \sum_{m=1}^{2l+1} P_{2l+1}^m(0)P_{2l+1}^m(\cos \alpha) \cos m \phi \right\}. \]

This looks ugly, but in fact when we plug it into the integrals (9) and (10) the \( \sin \phi \) and \( \cos \phi \) terms beat against the \( \cos m \phi \) term, integrating to 0 in the former case and \( \pi \delta_{m1} \) in the latter. The torque is

\[ N_x = 0 \]

\[ N_y = \frac{\pi \mu_0 I l b^2}{a} \sum_{l=0}^{\infty} \frac{(-1)^l(2l + 1)!!}{2^l l!} \left( \frac{b}{a} \right)^{2l} \frac{P_{l+1}^1(0)P_{l+1}^1(\cos \alpha)}{P_{l+1}^1(0)P_{l+1}^1(\cos \alpha)}. \]

To finish we just need to rewrite the numerical factor under the sum:

\[ \frac{(-1)^l(2l + 1)!!}{2^l l!} P_{2l+1}^1(0) = \frac{(2l + 1)!!}{2^l l!} \left[ \Gamma(l + 3/2) \Gamma(l + 1) \Gamma(3/2) \right] \]

\[ = \frac{(2l + 3 - 2)(2l + 3 - 4)(2l + 3 - 6) \cdots (5)(3)}{2^{l+1} \Gamma(l+1) \Gamma(3/2)} \frac{\Gamma(l + 3/2)}{\Gamma(l + 1) \Gamma(3/2)} \]

\[ = \left( \frac{l + 3/2}{\Gamma(l + 1) \Gamma(3/2)} \right)^2 \]

\[ = (l + 1)^2 \left[ \frac{\Gamma(l + 3/2)}{\Gamma(l + 2) \Gamma(3/2)} \right]^2 \]
So my answer is

$$N_y = \frac{\pi \mu_0 I^l b^2}{a} \sum_{l=0}^{\infty} (l+1)^2 \left[ \frac{\Gamma(l+3/2)}{\Gamma(l+2)\Gamma(3/2)} \right] \left( \frac{b}{a} \right)^{2l} P_{2l+1}^1(\cos \alpha).$$

Evidently I'm off by a factor of $1/(l+1)(2l+1)$ under the sum, but I can't find where. Can anybody help?

**Problem 5.13**

A sphere of radius $a$ carries a uniform surface-charge distribution $\sigma$. The sphere is rotated about a diameter with constant angular velocity $\omega$. Find the vector potential and magnetic-flux density both inside and outside the sphere.

**Problem 5.14**

A long, hollow, right circular cylinder of inner (outer) radius $a$ ($b$), and of relative permeability $\mu_r$, is placed in a region of initially uniform magnetic-flux density $B_0$ at right angles to the field. Find the flux density at all points in space, and sketch the logarithm of the ratio of the magnitudes of $B$ on the cylinder axis to $B_0$ as a function of $\log_{10} \mu_r$ for $a^2/b^2 = 0.5, 0.1$. Neglect end effects.

We'll take the cylinder axis as the $z$ axis of our coordinate system, and we'll take $B_0$ along the $x$ axis: $B_0 = B_0 \hat{i}$. To the extent that we ignore end effects, we may imagine the fields to have no $z$ dependence, so we effectively have a two-dimensional problem.

There are two distinct current distributions in this problem. The first is a current distribution $J_{\text{free}}$ giving rise to the uniform field $B_0$ far away from the cylinder; this current distribution is only nonvanishing at points outside the cylinder. The second is a current distribution $J_{\text{bound}} = \nabla \times M$ existing only within the cylinder. Since there is no free current within the cylinder or in its inner region, the equations determining $H$ in those regions are

$$\nabla \cdot B = \nabla \cdot (\mu H) = 0, \quad \nabla \times H = J_{\text{free}} = 0.$$

These imply that, within the cylinder and in its inner region, we may derive $H$ from a scalar potential: $H = -\nabla \Phi_m$, with $\Phi_m$ satisfying the Laplace equation.

In the external region, there is free current, so things are not so simple. To proceed we may separate the $H$ field in the external region into two components: one that arises from the free current, and one that arises from the bound currents within the cylinder. The former is just $(1/\mu_0)B_0$ and the second is again derivable from a scalar potential satisfying the Laplace equation. So, in the external region, $H = (1/\mu_0)B_0 - \nabla \Phi_m$. 
So our task is to find expressions for $\Phi_m$ in the three regions such that the boundary conditions on $B$ and $H$ are satisfied at the borders of the regions. Writing down the solutions of the 2-D Laplace equation in the three regions, and excluding terms which blow up as $\rho \to 0$ or $\rho \to \infty$, we have

$$\Phi_m(\rho, \phi) = \begin{cases} 
\sum_{n=1}^{\infty} \rho^n \left[ A_n \cos n\phi + B_n \sin n\phi \right] & r < a \\
\sum_{n=1}^{\infty} \rho^n \left[ C_n \cos n\phi + D_n \sin n\phi \right] + \rho^{-n} \left[ E_n \cos n\phi + F_n \sin n\phi \right] & a < r < b \\
\sum_{n=1}^{\infty} \rho^{-n} \left[ G_n \cos n\phi + H_n \sin n\phi \right] & r > b
\end{cases}$$

Actually, we may argue on symmetry grounds that the sin terms must all vanish: otherwise, the fields would take different values on the positive and negative $y$ axes, but there is nothing in the problem distinguishing these axes from each other. With this simplification we may write down expressions for the components of the $H$ field in the three regions:

$$H_r = \begin{cases} 
-\frac{\partial}{\partial r} \Phi_m = \sum_{n=1}^{\infty} -n A_n \rho^{n-1} \cos n\phi, & r < a \\
-\frac{\partial}{\partial r} \Phi_m = \sum_{n=1}^{\infty} -n \left( C_n \rho^{n-1} - E_n \rho^{-n+1} \right) \cos n\phi, & a < r < b \\
(1/\mu_0) B_0 r - \frac{\partial}{\partial r} \Phi_m = \left[ (1/\mu_0) B_0 \cos \phi + \sum_{n=1}^{\infty} n G_n \rho^{-n+1} \cos n\phi \right], & r < b
\end{cases}$$

$$H_\phi = \begin{cases} 
-\frac{\partial}{\partial \phi} \Phi_m = \sum_{n=1}^{\infty} n A_n \rho^{n-1} \sin n\phi, & r < a \\
-\frac{\partial}{\partial \phi} \Phi_m = \sum_{n=1}^{\infty} n \left( C_n \rho^{n-1} + E_n \rho^{-n+1} \right) \sin n\phi, & a < r < b \\
(1/\mu_0) B_0 \sin \phi - \frac{\partial}{\partial \phi} \Phi_m = \left[ - (1/\mu_0) B_0 \sin \phi + \sum_{n=1}^{\infty} n G_n \rho^{-n+1} \sin n\phi \right], & r < b
\end{cases}$$

The boundary conditions at $r = b$ are that $\mu H_r$ and $H_\phi$ be continuous, where $\mu = \mu_0$ outside the cylinder and $\mu_r \mu_0$ inside. With the above expressions for the components of $H$, we have

$$\frac{1}{\mu_0} B_0 \cos \phi + \sum_{n=1}^{\infty} n G_n b^{-(n+1)} \cos n\phi = \mu_r \sum_{n=1}^{\infty} -n \left( C_n b^{n-1} - E_n b^{-n+1} \right) \cos n\phi$$

$$-\frac{1}{\mu_0} B_0 \sin \phi + \sum_{n=1}^{\infty} n G_n b^{-(n+1)} \sin n\phi = \sum_{n=1}^{\infty} n \left( C_n b^{n-1} + E_n b^{-n+1} \right) \sin n\phi.$$

We may multiply both sides of these by $\cos n\phi$ and $\sin n\phi$ and integrate from
0 to $2\pi$ to find

\[
\frac{1}{\mu_0} B_0 + G_1 b^{-2} = -\mu_r C_1 + \mu_r E_1 b^{-2}
\]

(13)

\[
G_n b^{-(n+1)} = -\mu_r \left( C_n b^{n-1} - E_n b^{-(n-1)} \right), \quad n \neq 1
\]

(14)

\[
-\frac{1}{\mu_0} B_0 + G_1 b^{-2} = C_1 + E_1 b^{-2}
\]

(15)

\[
G_n b^{-(n+1)} = \left( C_n b^{n-1} + E_n b^{-(n+1)} \right), \quad n \neq 1
\]

(16)

Similarly, at $r = a$ we obtain

\[
A_1 = \mu_r C_1 - \mu_r E_1 a^{-2}
\]

(17)

\[
A_n a^{n-1} = \mu_r \left( C_n a^{n-1} - E_n a^{-(n+1)} \right), \quad n \neq 1
\]

(18)

\[
A_1 = C_1 + E_1 a^{-2}
\]

\[
A_n a^{n-1} = \left( C_n a^{n-1} + E_n a^{-(n+1)} \right), \quad n \neq 1.
\]

(19)

For $n \neq 1$, the only solution turns out to be $A_n = C_n = E_n = G_n = 0$. For $n = 1$, multiplying (15) by $\mu_r$ and adding and subtracting with (13) yields

\[
2\mu_r C_1 = -(\mu_r + 1) \frac{B_0}{\mu_0} + (\mu_r - 1) G_1 b^{-2}
\]

(20)

\[
2\mu_r E_1 = (1 - \mu_r) \frac{B_0}{\mu_0} b^2 + (\mu_r + 1) G_1.
\]

(21)

On the other hand, multiplying (18) by $\mu_r$ and adding and subtracting with (17) yields

\[
2\mu_r C_1 = (\mu_r + 1) A_1
\]

(22)

\[
2\mu_r E_1 = (\mu_r - 1) a^2 A_1.
\]

(23)

Equating (20) with (22), we find

\[
A_1 = -\frac{B_0}{\mu_0} + \frac{(\mu_r - 1)}{(\mu_r + 1)} G_1 b^{-2}
\]

while equating (21) with (23) yields

\[
A_1 = -\frac{B_0}{\mu_0} \left( \frac{b^2}{a^2} \right) + \frac{(\mu_r + 1)}{(\mu_r - 1)} G_1 a^{-2}
\]

and now equating these two equations gives

\[
G_1 = \left[ 1 - \left( \frac{a}{b} \right)^2 \right] \frac{(\mu_r^2 - 1) b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2} \left( \frac{B_0}{\mu_0} \right) b^2.
\]
The other coefficients may be worked out from this one:

\[ A_1 = \frac{-4\mu_r b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}\frac{B_0}{\mu_0} \]
\[ C_1 = \frac{-2(\mu_r + 1)b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}\frac{B_0}{\mu_0} \]
\[ E_1 = \frac{-2(\mu_r - 1)b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}\frac{B_0}{\mu_0}a^2. \]

The \( \mathbf{H} \) field is

\[
\mathbf{H} = \frac{4\mu_r b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}\frac{B_0}{\mu_0} \hat{\mathbf{i}}, \quad r < a
\]
\[
= \frac{2b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}\frac{B_0}{\mu_0} \left\{ (\mu_r + 1) + (\mu_r - 1) \left( \frac{a}{r} \right)^2 \hat{\mathbf{i}} - 2(\mu_r - 1) \left( \frac{a}{r} \right)^2 \cos \phi \hat{\mathbf{\phi}} \right\}, \quad a < r < b
\]
\[
= \frac{B_0}{\mu} + \frac{(b^2 - a^2)(\mu_r^2 - 1)}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2} \left( \frac{B_0}{\mu_0} \right) \left( \frac{b^2}{r^2} \right) \left[ \hat{\mathbf{i}} + 2 \sin \phi \hat{\mathbf{\phi}} \right], \quad r > b.
\]

The ratio \( r \) of the field within the cylinder to the external field is

\[
r = \frac{4\mu_r}{(\mu_r + 1)^2 - (\mu_r - 1)^2 \frac{a^2}{b^2}}.
\]

This relationship is graphed in Figure 2.
Problem 5.16

A circular loop of wire of radius $a$ and negligible thickness carries a current $I$. The loop is centered in a spherical cavity of radius $b > a$ in a large block of soft iron. Assume that the relative permeability of the iron is effectively infinite and that of the medium in the cavity, unity.

(a) In the approximation of $b \gg a$, show that the magnetic field at the center of the loop is augmented by a factor $(1 + a^3/2b^3)$ by the presence of the iron.

(b) What is the radius of the "image" current loop (carrying the same current) that simulates the effect of the iron for $r < b$?

(a) There are two distinct current distributions in this problem: the free current density $\mathbf{J}_1$ flowing in the loop, and the bound current density $\mathbf{J}_2$ flowing in the iron. These give rise to two fields $\mathbf{B}_1$ and $\mathbf{B}_2$, which must be summed at each point in space to get the observed field.

$\mathbf{B}_1$ is just the field of a planar current loop, which Jackson has already worked out for us in his section 5.5:

\[
\mathbf{B}_{1r} = \begin{cases} 
\frac{\mu_0 I}{2a} \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 1)!!}{2^n n!} \left(\frac{r}{a}\right)^{2n} P_{2n+1}(\cos \theta), & r < a \\
\frac{\mu_0 I a^2}{2r^3} \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 1)!!}{2^n n!} \left(\frac{a}{r}\right)^{2n} P_{2n+1}(\cos \theta), & r > a.
\end{cases}
\] (24)

\[
\mathbf{B}_{1\theta} = \begin{cases} 
\frac{\mu_0 I}{4a} \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1)!!}{2^{n-1} n!} \left(\frac{r}{a}\right)^{2n} P_{2n+1}^1(\cos \theta), & r < a \\
-\frac{\mu_0 I a^2}{4r^3} \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 1)!!}{2^n (n + 1)!} \left(\frac{a}{r}\right)^{2n} P_{2n+1}^1(\cos \theta), & r > a.
\end{cases}
\] (25)

On the other hand, since $\mathbf{J}_2$ vanishes for $r < b$, the field $\mathbf{B}_2$ to which it gives rise has no divergence or curl in that region, which means that throughout the region it may be derived from a scalar potential satisfying the Laplace equation:

\[
\mathbf{B}_2 = -\nabla \Phi_m = -\nabla \left[ \sum_{n=0}^{\infty} A_n r^n P_n(\cos \theta) \right]
\]

\[
\rightarrow B_{2r} = \sum_{n=1}^{\infty} n A_n r^{n-1} P_n(\cos \theta)
\] (26)

\[
B_{2\theta} = \sum_{n=1}^{\infty} A_n r^{n-1} P^1_n(\cos \theta)
\] (27)
Since the iron filling the space \( r > b \) is assumed to have infinite permeability, the \( \mathbf{H} \) field (and hence the \( \mathbf{B} \) field, since \( \mathbf{B} = \mathbf{H} \) for \( r < b \)) must be strictly radial at the boundary \( r = b \). The \( A_n \) coefficients are thus determined by the requirement that (27) and (25) sum to zero at \( r = b \):

\[
\sum_{n=1}^{\infty} A_n b^{n-1} P_n^1(\cos \theta) = \frac{\mu_0 I a^2}{4b^3} \sum_{n=0}^{\infty} \frac{(-1)^n(2n + 1)!!}{2^n(n + 1)!} \left( \frac{a}{b} \right)^{2n} P_{2n+1}^1(\cos \theta).
\]

The orthogonality of the associated Legendre polynomials requires that each term in the sum cancel individually, whence

\[
A_{2n} = 0
\]

\[
A_{2n+1} = \frac{\mu_0 I a^2}{4b^3} \frac{(-1)^n(2n + 1)!!}{2^n(n + 1)!} \left( \frac{a}{b^2} \right)^{2n}.
\]

Then the field of the bound current in the iron is determined everywhere in the region \( r < b \):

\[
B_{2r} = \frac{\mu_0 I a^2}{4b^3} \sum_{n=0}^{\infty} \frac{(-1)^n(2n + 1)(2n + 1)!!}{2^n(n + 1)!} \left( \frac{ar}{b^2} \right)^{2n} P_{2n+1}^1(\cos \theta) \quad (28)
\]

\[
B_{2\theta} = \frac{\mu_0 I a^2}{4b^3} \sum_{n=0}^{\infty} \frac{(-1)^n(2n + 1)!!}{2^n(n + 1)!} \left( \frac{ar}{b^2} \right)^{2n} P_{2n+1}^1(\cos \theta). \quad (29)
\]

As \( r \to 0 \), \( B_{2z} \to 0 \) and \( B_{2r} \to \mu_0 I a^2/4b^3 \), while \( B_{1r} \to \mu_0 I/2a \), so the total field at \( r = 0 \) is

\[
B_r(r = 0) = B_{1r}(r = 0) + B_{2r}(r = 0) = \frac{\mu_0 I}{2a} + \frac{\mu_0 I a^2}{4b^3} = \frac{\mu_0 I}{2a} \left[ 1 + \frac{a^4}{2b^3} \right].
\]

(b) The \( \mathbf{B}_2 \) field may be attributed to an image current ring outside \( r = b \) if, for suitable redefinitions of \( I \) and \( a \), the expressions (28) and (29) can be made to look like the \( r < a \) versions of (24) and (25).

**Problem 5.18**

A circular loop of wire having a radius \( a \) and carrying a current \( I \) is located in vacuum with its center a distance \( d \) away from a semi-infinite slab of permeability \( \mu \). Find the force acting on the loop when

(a) the plane of the loop is parallel to the face of the slab,

(b) the plane of the loop is perpendicular to the face of the slab.

(c) Determine the limiting form of your answer to parts a and b when \( d \gg a \). Can you obtain these limiting values in some simple and direct way?

(a) We’ll take the loop to be at \( z = +d \), and the slab of permeability \( \mu \) to occupy the space \( z < 0 \), so that the boundary surface is \( z = 0 \).
In the region \( z < 0 \), there is no free current, so \( \nabla \times \mathbf{H} = 0 \) everywhere; thus \( \mathbf{H} \) may be obtained from a scalar potential, \( \mathbf{H} = -\nabla \Phi_m \), and since \( \nabla \cdot \mathbf{H} = 0 \) as well we have \( \nabla^2 \Phi_m = 0 \). The azimuthally symmetric solution of the Laplace equation in cylindrical coordinates that remains finite as \( z \to -\infty \) is

\[
\Phi_m(z < 0) = \int_0^\infty dk \ A(k) e^{kz} J_0(k\rho),
\]

and from this we obtain

\[
H_\rho(z < 0) = -\frac{\partial}{\partial \rho} \Phi_m = -\int_0^\infty dk \ k A(k) e^{kz} J_0'(k\rho)
= \int_0^\infty dk \ k A(k) e^{kz} J_1(k\rho)
\]

\[
H_z(z < 0) = -\frac{\partial}{\partial z} \Phi_m = -\int_0^\infty dk \ k A(k) e^{kz} J_0(k\rho).
\]

On the other hand, for \( z > 0 \) we may decompose the \( \mathbf{H} \) field into two components: one component \( \mathbf{H}_1 \) arising from the current loop, and a second component \( \mathbf{H}_2 \) arising from the bound currents running in the slab. \( \mathbf{H}_1 \) is just given by the curl of the vector potential we worked out in Problem 5.10:

\[
\mathbf{H}_1 = \frac{1}{\mu_0} \nabla \times \mathbf{A}, \quad \mathbf{A} = A_\phi \hat{\phi}, \quad A_\phi = \begin{cases} \frac{\mu_0 I a}{2} \int_0^\infty dk \ k e^{-k(z-d)} J_1(ka) J_1(k\rho), & z > d \\ \frac{\mu_0 I a}{2} \int_0^\infty dk \ k e^{-k(d-z)} J_1(ka) J_1(k\rho), & z < d \end{cases},
\]

so

\[
H_{1\rho} = -\frac{1}{\mu_0} \frac{\partial}{\partial \rho} A_\phi
= \begin{cases} \frac{I a}{2} \int_0^\infty dk \ k e^{-k(z-d)} J_1(ka) J_1(k\rho), & z > d \\ -\frac{I a}{2} \int_0^\infty dk \ k e^{-k(d-z)} J_1(ka) J_1(k\rho), & z < d \end{cases},
\]

\[
H_{1z} = \frac{1}{\mu_0 \rho} \frac{\partial}{\partial \rho} (\rho A_\phi)
= \begin{cases} \frac{I a}{2} \int_0^\infty dk \ k e^{-k(z-d)} J_1(ka) \left[ \frac{1}{k \rho} J_1(k\rho) - J_0(k\rho) \right], & z > d \\ \frac{I a}{2} \int_0^\infty dk \ k e^{-k(d-z)} J_1(ka) \left[ \frac{1}{k \rho} J_1(k\rho) - J_0(k\rho) \right], & z < d \end{cases}.
\]

In the last two equations we may use Jackson’s identity (3.87),

\[
\frac{1}{k \rho} J_1(k\rho) = \frac{1}{2} [J_0(k\rho) + J_2(k\rho)]
\]
to rewrite $H_{1z}$ as

$$H_{1z} = \begin{cases} 
\frac{Ia}{4} \int_0^\infty dk e^{-k(d-z)} J_1(ka) [J_2(kp) - J_0(kp)], & z > d \\
\frac{Ia}{4} \int_0^\infty dk e^{-k(d-z)} J_1(ka) [J_2(kp) - J_0(kp)], & z < d.
\end{cases} \quad (35)$$

Since the $H_2$ field arises entirely from bound currents, it may also be derived from a scalar potential $\Phi_m$ satisfying the Laplace equation. The azimuthally symmetric solution of the Laplace equation in cylindrical coordinates that remains finite for all $\rho$ and as $z \to +\infty$ is

$$\Phi_m(z > 0) = \int_0^\infty dk B(k) e^{-kz} J_0(k\rho)$$

and the components of $H_2$ are

$$H_{2r}(z > 0) = -\int_0^\infty dk k B(k) e^{-kz} J_1(k\rho) \quad (36)$$

$$H_{2z}(z > 0) = \int_0^\infty dk k B(k) e^{-kz} J_0(k\rho). \quad (37)$$

The required forms of the functions $A(k)$ and $B(k)$ are determined by the boundary conditions on $H$ at the medium boundary, $z = 0$:

$$H_{\rho}(z = 0_-) = H_{\rho}(z = 0_+) \quad \mu H_{\rho}(z = 0_-) = \mu_0 H_{\rho}(z = 0_+).$$

Equating (32) with the sum of (??) and (??), we have

$$-\int_0^\infty dk k A(k) J_0(k\rho) = \frac{\mu_0 Ia}{2} \int_0^\infty dk k e^{-kd} J_1(ka) (J_2(kp) - J_0(kp)) + \int_0^\infty dk k B(k) J_0(k\rho)$$