Master Equations for Two-Component Matching Networks

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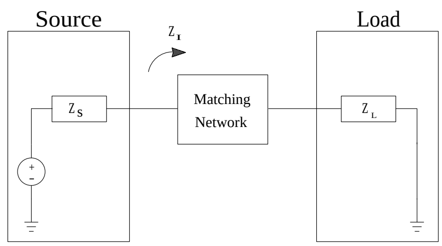
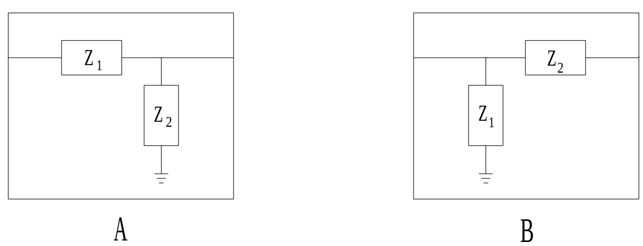
The *impedance matching* problem is illustrated below. We have a driving source with output impedance Z_S, and a load of impedance Z_l, and we want to insert a matching network between the two to ensure maximum power transfer to the load.



The maximum-power-transfer condition is met when Z_I (the impedance seen looking into the matching network from the source) equals the complex conjugate of the source impedance Z_S.

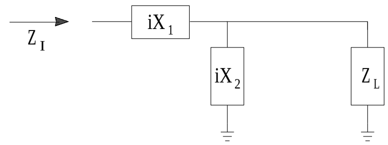
A two-component matching network can assume one of two forms:



where the Z elements inside the boxes are lossless elements (i.e. inductors or capacitors) to avoid power dissipation in the matching network.

It is instructive to illustrate the calculation of the impedances Z_1 and Z_2 needed to obtain a match between given source and load impedances. In what follows we will do this for each of the two possible topologies.

Topology A. Here the situation is as follows:



where we have written iX_1 and iX_2 for the Z elements since they are pure reactances. The impedance Z_i seen looking into the whole thing is

$$\begin{aligned} Z_I &= iX_1 + (iX_2 || Z_L) && (1)\\ &= iX_1 + \frac{iX_2 Z_L}{iX_2 + Z_L} && (2)\\ &= iX_1 + \frac{iX_2(R_L + iX_L)}{iX_2 + R_L + iX_L} && (3) \end{aligned}$$

and the idea is to make this equal to the conjugate of our source impedance, i.e.

$$Z_I = R_S - iX_S. \tag{4}$$

Substituting (3) into (4) we find

$$R_S - iX_S = iX_1 + \frac{iX_2(R_L + iX_L)}{iX_2 + R_L + iX_L} \tag{5}$$

or

$$\begin{aligned} &iR_S X_2 + R_S R_L + iR_S X_L + X_S X_2 - iX_S R_L + X_S X_L = \\ &- X_1 X_2 + iX_1 R_L - X_1 X_L + iX_2 R_L - X_2 X_L. \end{aligned}$$

Equating the real and imaginary parts of (6) we find

$$\begin{aligned} R_S R_L + X_S X_2 + X_S X_L &= -X_1 X_2 - X_1 X_L - X_2 X_L && (6)\\ R_S X_2 + R_S X_L - X_S R_L &= +X_1 R_L + X_2 R_L. && (7) \end{aligned}$$

We have two equations and two variables, so the system is solvable.

Starting first with (7), we find

$$X_1 R_L = R_S X_2 - R_L X_2 + R_S X_L - R_L X_S$$

and dividing through by R_L gives

$$\begin{aligned} X_1 &= \frac{R_S}{R_L} X_2 - X_2 + \frac{R_S}{R_L} X_L - X_S \\ &= (\gamma - 1) X_2 + \gamma X_L - X_S && (8) \end{aligned}$$

where we have defined

$$\gamma = \frac{R_S}{R_L}. \tag{9}$$

Now substituting (8) into (6) we find

$$R_S R_L + X_S X_2 + X_S X_L =$$
$$-(\gamma - 1)X_2^2 - \gamma X_2 X_L + X_S X_2 - (\gamma - 1)X_2 X_L - \gamma X_L^2 + X_L X_S - X_2 X_L.$$

Happily, there is some cancellation here. We obtain

$$(\gamma - 1)X_2^2 + (2\gamma X_L)X_2 + \gamma(X_L^2 + R_L^2) = 0$$

and now use the quadratic formula:

$$\begin{aligned} X_2 &= \frac{-2\gamma X_L \pm \sqrt{4\gamma^2 X_L^2 - 4(\gamma-1)\gamma(X_L^2 + R_L^2)}}{2(\gamma - 1)} \\ &= -\frac{\gamma}{\gamma - 1} X_L \pm \frac{[\gamma(R_L^2 + X_L^2 - R_L R_S)]^{1/2}}{\gamma - 1}. \end{aligned} \tag{10}$$

Equation (10), together with (9) and (8), gives values for X_1 and X_2 necessary for a conjugate match from Z_S to Z_L. To design an actual matching network, you would apply these formulas to calculate X_1 and X_2, and then realize X_1 and X_2 with inductors or capacitors chosen to have the given impedances at your desired operating frequency. For example, X_1 is realized by

an inductor of value	$X1/2\pi f$	henries,	if	$X1 > 0$
a capacitor of value	$2\pi f \|\|X1\|\|$	farads,	if	$X1 < 0$.

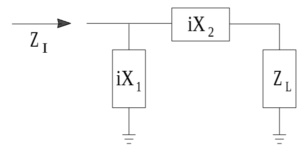
Here f is the frequency, in Hertz, at which you want the match to work.

Note that equation (10) only gives usable physical results if the argument of the square root is nonnegative, i.e. if

$$R_L^2 + X_L^2 > R_L R_S. \tag{11}$$

This condition is not difficult to understand physically. First note that, if the load is a pure resistance ($X_L = 0$), the condition reduces to $R_L > R_S$. And, indeed, in this topology what we're doing is *shunting* the load impedance with X_2, then adding to it a series reactance X_1, which only modifies the imaginary part. So the effect of the match is to *reduce* the real part of the load impedance, which can only be what we want to do if the real part of the load is greater than that of the source. However, if the load has a reactive component ($X_L \neq 0$), then the shunting effect of X_2 can actually *increase* the real component of the load impedance. In this case this match can still work even is $R_L < R_S$, provided X_L is large enough to satisfy (11).

Topology B. Here the situation is as follows:



The impedance looking in to the matching network is now

$$Z_I = iX_1 || (iX_2 + Z_L) \tag{12}$$

and we are to make this equal to $Z_S^* = R_S - iX_S$. The algebra is exactly analogous to that for topology A, and we will not reproduce it here. The result is

$$X_1 = \frac{\gamma'}{\gamma' - 1} X_S \pm \frac{[\gamma'(R_S^2 + X_S^2 - R_L R_S)]^{1/2}}{\gamma' - 1} \tag{13}$$

$$X_2 = (\gamma' - 1)X_1 - X_L + \gamma' X_S \tag{14}$$

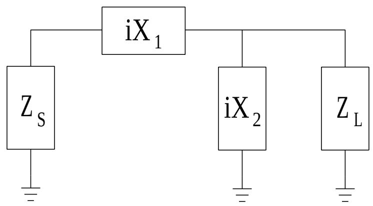
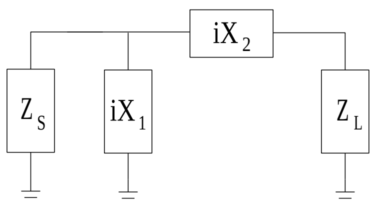
$$\gamma' = \frac{R_L}{R_S}. \tag{15}$$

Note that γ' is the inverse of γ as defined previously. In other respects these equations are similar to their counterparts (8), (9), and (10). The condition for (13) to give physical results is

$$R_S^2 + X_S^2 > R_L R_S \tag{16}$$

Again we can understand this result physically: with this topology we're actually shunting the *source* impedance, so we would expect this match to work only if the source impedance is greater than the load impedance. Indeed, if the source has no reactive component then (16) reduces to just this condition. On the other hand, if $X_S \neq 0$, then this match can still work even if the real component of the source impedance is *smaller* than the real component of the load impedance.

Summary The results of the above considerations are summarized in the following table.

Topology	Equations	Condition
	$X_2 = -\frac{\gamma}{\gamma - 1}X_L \pm \frac{[\gamma(R_L^2 + X_L^2 - R_L R_S)]^{1/2}}{\gamma - 1}$ $X_1 = (\gamma - 1)X_2 + \gamma X_L - X_S$ $\gamma = \frac{R_S}{R_L}$	$R_L^2 + X_S^2 > R_L R_S$
	$X_1 = -\frac{\gamma}{\gamma - 1}X_S \pm \frac{[\gamma(R_S^2 + X_S^2 - R_L R_S)]^{1/2}}{\gamma - 1}$ $X_2 = (\gamma - 1)X_1 + \gamma X_S - X_L$ $\gamma = \frac{R_L}{R_S}$	$R_S^2 + X_S^2 > R_L R_S$